The Islamic University of Gaza
Deanery of Graduate Studies
Faculty of Electrical Engineering
Control Systems Department

الجامعة الإسلامية-غزة
عمادة الدراسات العــليا
كلية الهندسة الكهربائية
قــسم أنظمة الـتـحكــم

Solving Quadratic Optimal Control Problems Using Legendre Scaling Function and Iterative Technique

By

Eng. Mohammed J. M. Wadi

Advisors

Dr. Hatem Elaydi | Dr. Hussein Jaddu

"A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Electrical Engineering "

Gaza-Palestine

1432-2011

Thesis approval

Abstract

During the last three decades several approximation techniques based on the property of functions orthogonality were proposed to solve different classes of optimal control problems (OCP_S).

The methods used to solve OCP_S are classified into two types: the direct methods are discretization and parameterization while indirect methods are Caley-Hamilton and Euler-Lagrange. The direct parameterization methods are classified into three ways control parameterization, state parameterization, and state-control parameterization.

The proposed method in this thesis uses state-control parameterization via Legendre scaling function in which OCP_S is converted into quadratic programming. In addition, when OCP in quadratic form, it is easy to solve it by using any software package like MATLAB, Mathmatica, or Maple.

The optimal control problems investigated in this thesis deals with linear time invariant (LTI) systems, linear time varying (LTV) systems, and nonlinear systems.

The LTI and LTV problems were parameterized based on the Legendre scaling function such that the cost function and the constraints are casted in terms of state and control parameters while, complex nonlinear OCP_S can be solved by proposed method after converted to a sequence of time varying problem using iterative technique.

To demonstrate applicability and effectiveness of the proposed technique various numerical examples are solved and the results are better when compared with other methods.

ملخص :

حل مسائل التحكم التربيعي الأمثل باستخدام دالة اللجندر المقاسة والتقنية التكرارية

اقترحت أثناء العقود الثلاثة الأخيرة مجموعة كبيرة من طرق التقريب المعتمدة على خاصية التعامد في الدوال وذلك لحل مسائل التحكم الأمثل المختلفة.

الطرق المستخدمة في حل أسئلة التحكم الأمثل تصنف إلي قسمين الأولى :الطرق المباشرة وهي التجزئة والتقريب بينما تنقسم الطرق غير المباشرة إلى كالي-هاملتون و أولير-لاجرانج . طرق التقريب المباشرة تقسم إلى ثلاثة أقسام وهي : تقريب التحكم و تقريب الحالة وأخيرا تقريب الحالة والتحكم.

الطريقة المقترحة في هذه الأطروحة تستخدم النوع الثالث من طرق التقريب المباشرة حيث يتم تحويل مسألة التحكم الأمثل إلى مسألة تربيعية بواسطة دالة اللجندر المقاسة والتي تستخدم في تقريب الحالة والتحكم حيث يسهل حلها باستخدام أي برنامج يستخدم في حل مثل هذه المسائل مثل برنامج الماتلاب أو المثمتيكا أو المابل.

مسائل التحكم التي تناولتها هذه الأطروحة هي المسائل الغير معتمدة على الزمن و المعتمدة على الزمن بالإضافة إلى المسائل اللاخطية.

لحل المسائل الغير معتمدة على الزمن أو المعتمدة على الزمن يتم تقريب دالة الأداء والقيود باستخدام دالة اللجندر المقاسة مباشرة بينما المسائل اللاخطية يتم حلها بعد تحويلها إلى مسائل معتمدة على الزمن باستخدام التقنية التكرارية.

تم حل مجموعة من مسائل التحكم الأمثل وذلك لتوضيح مدى فعالية وتطبيق الطريقة المقترحة وكانت النتائج أفضل أو قريبة من نتائج الطرق الأخرى.

Dedication

To my parents, the soul of my brother Lotfy,

My wife and my sweet kids,

Jehad and Muhab

Acknowledgements

First and forever, all praise and thanks for Allah, who gave me the strength, and patience to carry out this work in this good manner.

I would like to deeply thank my supervisors Dr. Hatem Elaydi and Dr. Hussein Jaddu for their assistance, guidance, support, patience, and encouragement.

I would like to deeply thank my discussion committee Dr. Basil Hamad and Dr. Assad Abu-Jasser for their assistance and encouragement.

I would like to thank the Islamic University of Gaza for their help and support.

In addition, I would like to acknowledge the Academic staff of Engineering, who support me to carry out this work.

Finally, great thanks to my family for their endless praying and continuous support.

Contents

List of Tables

List of Figures

List of Abbreviations

OCP	Optimal Control Problem.
LTI	Linear Time Invariant.
LTV	Linear Time Varying.
TPBVP	Two-Point Boundary Value Problem.
HJBE	Hamilton-Jacobi-Bellman Equation.
ELE	Euler Lagrange Equation.
LSF	Legendre Scaling Function.
OMI	Operational Matrix of Integration.
MATLAB	Matrix Laboratory.
POM	Product Operational Matrix.
Poly	Polynomial.
Approx	Approximation.

Chapter One

Introduction

1.1 Background

Most of the computing techniques for the solution of nonlinear problems depend on approximating the solutions of nonlinear systems with linear systems in small regions of phase space. In general, nonlinear systems can't be solved obtaining general expressions; this motivated researchers to solve these nonlinear problems using numerical solutions i.e. optimal or suboptimal solutions.

Optimal control is an important science that deals with nonlinear optimal control problem (OCP) and the main objective of optimal control is to find an optimal controller that can be applied to the nonlinear system and to extrmize a certain cost function within the system's physical constraints.

Generally, optimal control can be classified as direct and indirect methods. Indirect methods are based on converting OCP into two-point boundary value problem(TPBVP), then solving the last one by Euler Lagrange technique or Hamilton-Jacobi-Bellman equation (HJBE). Direct methods can be implemented using discretization or parameterization methods.

1.2 Motivations

Approximation is one of the most successful applications among different applications of wavelets functions; thus a great number of researchers have tried to solve OCP_S using those functions.

The task of finding optimal controller $u^*(t)$ for nonlinear problems by indirect methods is often very difficult. Moreover, there are many disadvantages of indirect methods such as: (1) difficultly to obtain exact solution of nonlinear OCP_S using Euler-Lagrange or HJBE; (2) increasing problem computation by using artificial costates λ_{is}; (3) complete knowledge of system model is needed. In addition, there are many advantages of direct methods over indirect methods such as: (1) there is no need to use costates variables λ_{is}; (2) direct methods convert dynamic OCP into a static optimization problem. Thus, researchers are encouraged to use direct parameterization methods which are based on orthogonal functions and polynomials.

Researchers used direct parameterization methods which are classified into three types: state, control, and state-control parameterization [1-4].

Legendre scaling function (LSF) can be used for solving OCP_S because:

- It is not supported on the whole interval $(a \leq t < b)$, so it is effective in approximating functions with discontinuities or sharp changes.
- It has a closed form, so it is easy to obtain operational matrix of integration (OMI).
- It is not only orthogonal but also orthonormal.
- Rapid convergence.
- Small number of scaling parameter (K) and low order of Legendre polynomials (M) are needed to obtain very satisfactory results.
- Its OMI in the form of tridiagonal.

Thus, this work will be based on direct methods using state-control parameterization via LSF to solve quadratic OCP_S.

1.3 Thesis Objectives

Objectives of this thesis can be summarized as follows:

- Using LSF to approximate the state and control variables of OCP_S to solve linear time invariant and linear time-varying $OCP_{S.}$
- Solving complex nonlinear OCP_S by applying the iterative technique developed by Tomas and Banks [5-7] to convert the nonlinear quadratic OCP into a sequence of time varying quadratic OCP_S.
- Comparing between the proposed method with the other methods to show effectiveness of the proposed method.

1.4 Statement of Problem

The optimal control problem can be stated as: find an open loop optimal controller $u^*(t)$ or a closed loop optimal controller $u^*(x(t), t)$ that minimizes the following performance index

$$J = H\big(x(t_f), t_f\big) + \int_{t_o}^{t_f} G(x(t), u(t), t)\, dt \tag{1.1}$$

Subject to the system constraints

$$\dot{x} = f(x(t), u(t), t) \quad , x(t_o) = x_o \tag{1.2}$$

Where $t \in [t_o, t_f]$, $x \in R^n$ is the vector of states, $u \in R^m$ is the vector of controls, f is assumed continuous differentiable function with respect to all its arguments,while H and G are scalar functions.

Moreover, many techniques have been presented to solve OCP (1.1)-(1.2), and these methods can be classified into three methods as follows:

1. Calculus of variation (Euler Lagrange Equation).
2. Dynamic programming (Hamilton-Jacobi-Bellman Equation).
3. Nonlinear programming (Parameterization and discretization).

Due to the advantages of direct methods and the drawbacks of indirect methods, this thesis will based on parameterization method which is one techniques of nonlinear programming approaches. Figure (1.1) illustrates these computation methods

1.5 Literature Review

The solution of nonlinear OCP_S with constraints is very difficult, especially, when using indirect methods like HJBE. However, direct methods shown to be very useful and efficient in solving nonlinear OCP_S.

- Razzaghi and Yousefi [8] solved OCP_S using direct numerical methods; but handled only the inequality constraints and the equality constraints were linear depending on Legendre wavelets and Gauss node integration formula.
- Razzaghi and Yousefi [9] also proposed method for solution of nonlinear problems depending on calculus of variations in which only the performance index was nonlinear and the constraints were linear.
- Dadkhah and et al.[10] proposed numerical solution only for nonlinear Fredholm-Volterra integro-differential equation using Legendre wavelets.
- Sadek and et al.[11] proposed method of solving nonlinear OCPS based on modal space and Legendre wavelets; but their method was very complex.
- Babolian and Fattahzadeh. [12] proposed numerical method of solving differentiable equations using Chebyshev wavelets; but this method didn't take into account the nonlinear constraints.
- Jaddu and Vlach [13] proposed an approach to solve linear OCPS using Legendre wavelets.
- Jaddu[14] also proposed method for solving linear time-varying OCPS using Legendre wavelets, in which the time-varying problem converted to quadratic programming problem.
- Majdalawi[15] proposed a method for solving nonlinear OCPS using Legendre polynomial and state parameterization; but the main disadvantage that Legendre polynomial supported the whole interval and this will give poor results comparing with others methods.

The proposed method is based on Legendre scaling function and iterative technique where nonlinear OCP_S is solved. In addition, the performance index of optimal problem is in quadratic form and subject to different kind of state constraints.

1.6 Thesis Contributions

- ✓ Introducing a new method for solving nonlinear OCP_S using Legendre scaling function and iterative technique to provide a straight forward and convenient approach for digital computation.
- ✓ Providing numerical technique to solve linear OCP_S subject to state constraints.
- ✓ Presenting effective method to solve time varying OCP_S .
- ✓ Using simpler form of OMI that simplifies computations of optimal control.

- ✓ Presenting the property of multiplication of two Legendre scaling function which help solving time varying and nonlinear OCP_S.
- ✓ Keeping the performance index in the same format.

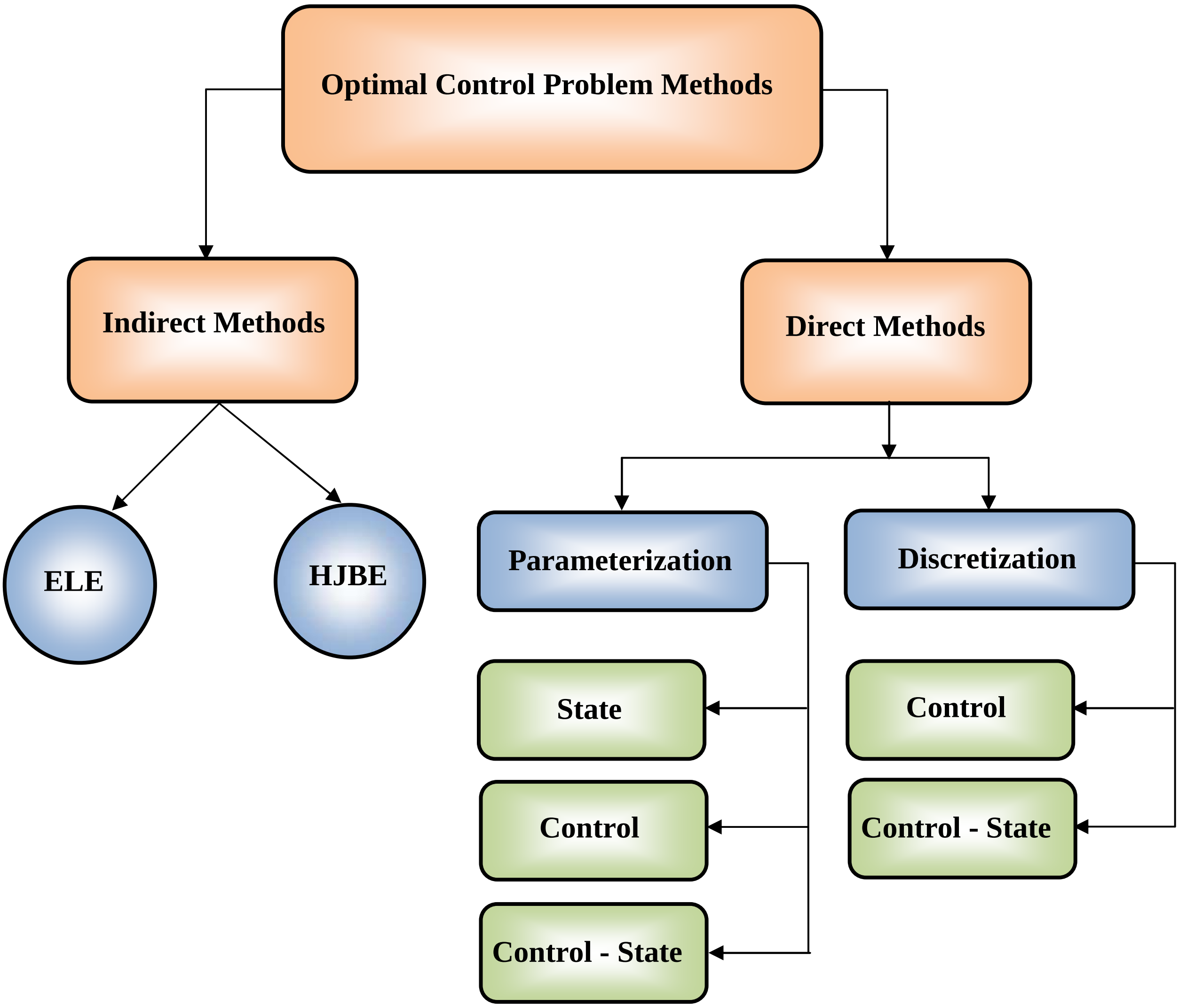

Figure (1.1): Computation Methods of Optimal Control Problems

1.7 Thesis Organization

The thesis is organized as follow:

Chapter Two is introduction to wavelets theory and Legendre scaling function. Chapter Three introduces linear quadratic OCPS and the Bolza form of the performance index. State and control variables parameterization via Legendre scaling function are also presented in Chapter Three. Chapter Four provides effective technique for solving time-varying OCP_S. In addition, the property of multiplication of two Legendre scaling functions is introduced. Chapter Five describes a method for solving unconstrained nonlinear OCP_S by converting nonlinear OCP_S into a sequence of linear time-varying OCP_S using iterative technique and two examples are shown. Chapter Six concludes this thesis.

Chapter Two

Introduction to Wavelets and Legendre Scaling Function

2.1 Introduction

Wavelets theory is a relatively new area in mathematical research which has considerable attention in a wide range of applications and engineering. In addition, wavelets are mathematical functions that separate data into different frequency components, and then present each component with a resolution matched to its scale. They have advantages over traditional Fourier methods in analyzing physical situations where the signal contains discontinuities and sharp spikes. Wavelets were developed independently in the fields of mathematics, quantum physics, and electrical engineering.

Moreover, wavelets emerge in the area of parameterization; due to their rapid convergence and their fast numerical algorithms.

There are a multitude of wavelets with different properties. It is important to choose the one with appropriate properties for a given application.

Most important properties are:

- ✓ The compact support property.
- ✓ The property of symmetry.
- ✓ Accuracy of approximation; particularly, with discontinuities functions.
- ✓ Their smoothness and regularity.
- ✓ Wavelets not supported on whole interval $(a \leq t < b)$.
- ✓ Orthogonality property.

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation (scaling) parameter a and the translation (shifting) parameter b vary continuously, we have the following family of continuous wavelets as [8]

$$\psi_{a,b}(t) = |a|^{-1/2}\, \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, a \neq 0 \tag{2.1}$$

Here, we would like to distinguish between wavelet and scaling functions. Wavelets are defined in the time domain by the wavelet function $\psi(t)$ which is called the mother wavelet and scaling function $\phi(t)$ which is called the father wavelet. Moreover, wavelet function has only time domain representation.

2.2 Legendre Scaling Function

In this section, we will introduce the definition, basis, and plots of Legendre scaling function. These basis will be the basic of our work in the following chapters. Legendre scaling function can be defined as in [13] as follows:

$$\Phi_{nm}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\, 2^{K/2}\, P_m(2^K t - 2n + 1) & for \quad \frac{2n-2}{2^K} \leq t < \frac{2n}{2^K} \\ 0 \quad otherwise & \end{cases} \tag{2.2}$$

Where P_m is the Legendre polynomial of order m; n refers to the section number,$n = 1,2,\ldots,2^{K-1}$; K is the scaling parameter and can assume any positive integer and $t \in [0,1]$.

Legendre polynomial can be defined as in [16] as follows:

$$P_m(x) = \frac{1}{2^m\, m!}\frac{d^m}{dx^m}(x^2-1)^m \tag{2.3}$$

From which the first four Legendre polynomial can be given

$$\begin{gathered} P_0(x) = 1 \\ P_1(x) = x \\ P_2(x) = \frac{1}{2}(3x^2-1) \\ P_3(x) = \frac{1}{2}(5x^3-3x) \end{gathered} \tag{2.4}$$

Now by using equations (2.2) and (2.4) and choose $M = 3$ and $K = 2$, then the four basis Legendre scaling functions can be given for $n = 1$ as follows

$$\left\{\begin{array}{l} \varphi_{10}(t) = \sqrt{2} \\ \varphi_{11}(t) = \sqrt{6}(4t-1) \\ \varphi_{12}(t) = \sqrt{\frac{5}{2}}[3(4t-1)^2-1] \\ \varphi_{13}(t) = \sqrt{\frac{7}{2}}[5(4t-1)^3-3(4t-1)] \end{array}\right\} \quad 0 \leq t < \frac{1}{2} \tag{2.5}$$

and for $n = 2$ as follows

$$\left.\begin{cases} \varphi_{20}(t) = \sqrt{2} \\ \varphi_{21}(t) = \sqrt{6}(4t-3) \\ \varphi_{22}(t) = \sqrt{\dfrac{5}{2}}\left[3(4t-3)^2 - 1\right] \\ \varphi_{23}(t) = \sqrt{\dfrac{7}{2}}\left[5(4t-3)^3 - 3(4t-3)\right] \end{cases}\right\} \quad \frac{1}{2} \leq t < 1 \qquad (2.6)$$

The basic functions of equations (2.5) and (2.6) can be plotted as shown in Figure (2.1) where the symmetry property is realized. In addition, this property simplifies the problem computation.

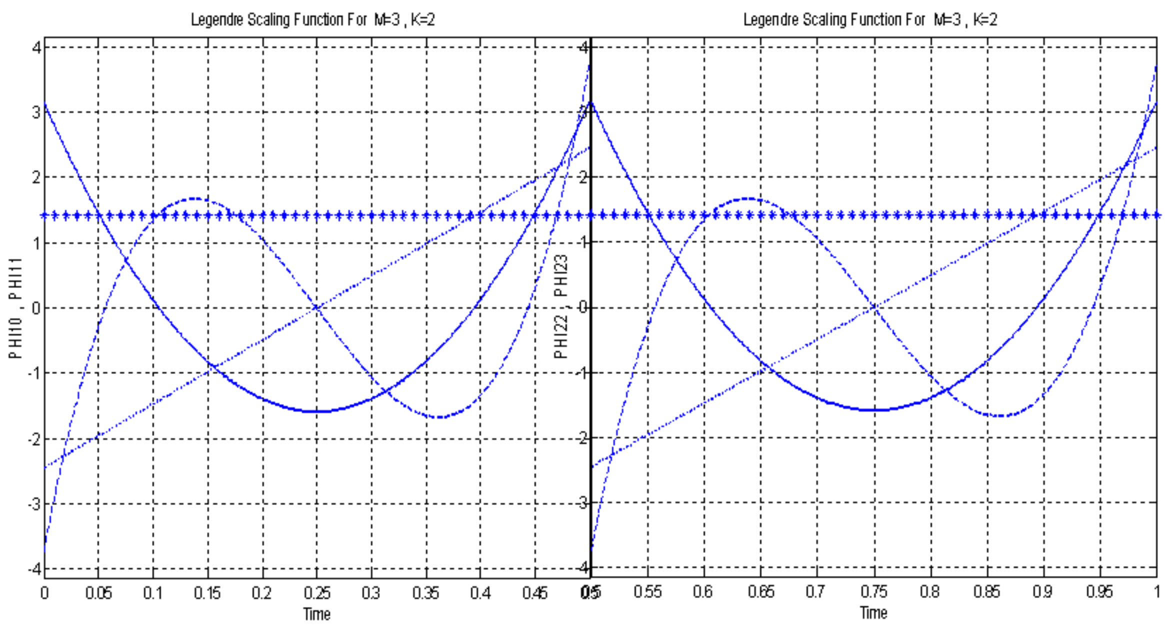

Figure (2.1): Legendre Scaling Functions Basis for $n = 1$ and $n = 2$

2.3 Approximation via Legendre Scaling Function

Any function $f(t)$, which is defined on the interval $[0,1]$, can be expanded using Legendre scaling function as follows :

$$f(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M} c_{nm}\, \phi_{nm}(t) \qquad (2.7)$$

Where

$$c_{nm} = \int_{\frac{2n-2}{2^K}}^{\frac{2n}{2^K}} f(t)\ \phi_{nm}(t)\ dt$$

$$C^T = \left[\, c_{10}\ c_{11} \dots\ c_{1M}\ c_{20} \dots c_{2M}\ c_{2^{K-1}0} \dots\ c_{2^{K-1}M}\right]$$

$$\Phi(t) = \left[\, \phi_{10}\ \phi_{11} \dots\ \phi_{1M}\ \phi_{20} \dots \phi_{2M}\ \phi_{2^{K-1}0} \dots\ \phi_{2^{K-1}M}\right]^T$$

The formula of equation (2.7) will be used in the next chapters to parameterize both state and control variables.

To show the effectiveness of Legendre scaling function in approximation, we introduce a square wave to be approximated as shown in Figures (2.2)-(2.4).

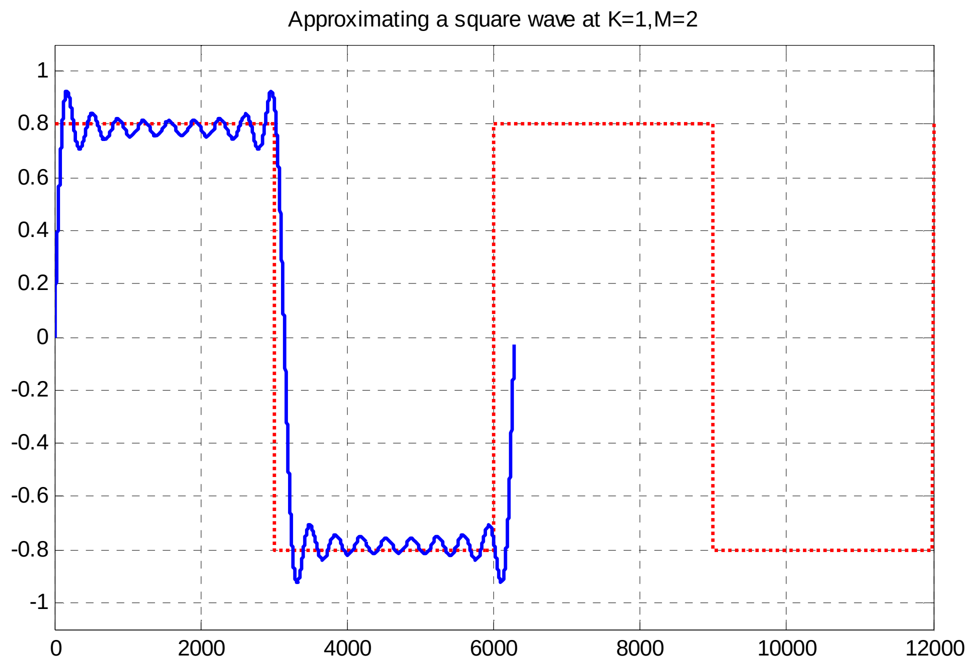

Figure (2.2): Approximating Square Wave at $K = 1$ and $M = 2$

When Figures (2.2), (2.3) and (2.4) are compared, we notice that increasing K and M will enhance the approximation. More over, to obtain the same accuracy as in Figure (2.4) using Fourier series the order of Fourier series will be nearly fifteen; so this illustrate the efficiency of Legendre scaling function in approximation.

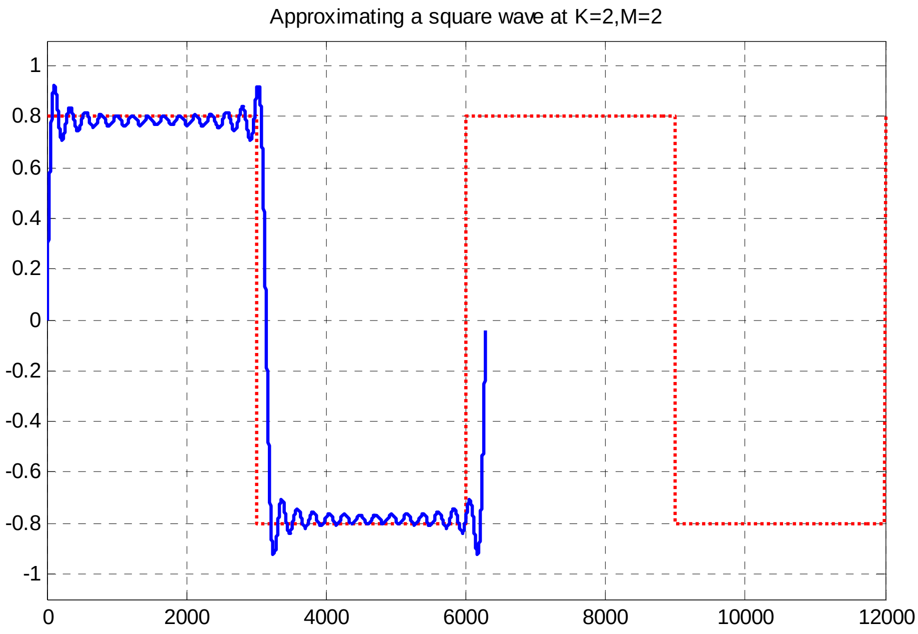

Figure (2.3): Approximating Square Wave at $K = 2$ and $M = 3$

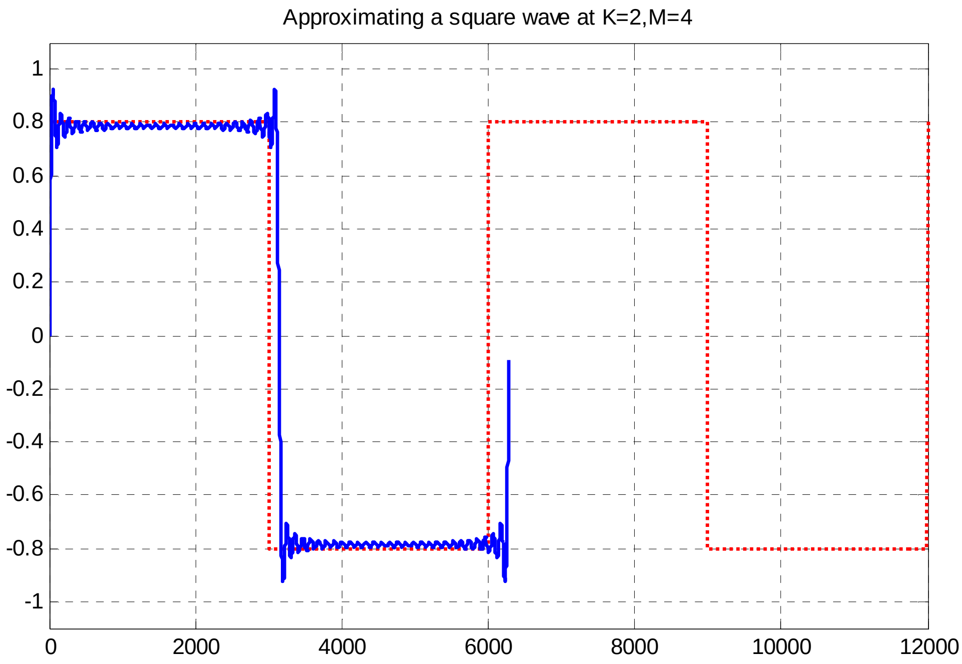

Figure (2.4): Approximating Square Wave at $K = 2$ and $M = 4$

Chapter Three

Linear Time Invariant Quadratic Optimal Control Problem

3.1 Introduction

In this chapter, linear approach is introduced to solve linear quadratic OCP_S; this step is necessary for the next chapters to solve time-varying and nonlinear OCP_S.

The main goal of the proposed method is to convert the nonlinear OCP_S into sequence of time varying OCP_S using iterative technique, then the optimal problem is converted to quadratic programming problem, which can be easily solved by any software package like MATLAB.

In this chapter, a new technique will be presented for handling linear quadratic control problems using state-control parameterization via Legendre scaling function.

3.2 Problem Statement

The optimal control problem can be considered as finding the optimal controller $u^*(t)$ that minimize a performance index such

$$J = \int_0^{t_f} (x^T Q x + u^T R u)\, dt \tag{3.1}$$

Subject to linear constraints and initial condition

$$\dot{x} = A\, x(t) + B\, u(t)\,, x(0) = x_o \tag{3.2}$$

Where $t \in [0, t_f]$, $x\,, x_o \in R^n$, $u \in R^m$, A, B are $n \times n$ and $n \times m$ constant matrices respectively. Q *is an* $n \times n$ positive semidefinite matrix and R *is an* $m \times m$ positive definite matrix.

3.3 Approximation via Legendre Scaling Function

State and control variables can be approximated using Legendre scaling function as follows

$$x_i(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M} a_{nm}^i\, \phi_{nm}(t) \quad i = 1,2, \dots, s \tag{3.3}$$

$$u_i(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M} b_{nm}^i\, \phi_{nm}(t) \quad i = 1,2, \dots, r \tag{3.4}$$

These equations can be written in compact form as follows

$$x(t) = \left(\mathbf{I_s} \otimes \phi^T(t)\right)\boldsymbol{a} \tag{3.5}$$

$$u(t) = \left(\mathbf{I_r} \otimes \phi^T(t)\right)\boldsymbol{b} \tag{3.6}$$

Where $\mathbf{I_s}, \mathbf{I_r}$ are identity matrices of dimension $s \times s$ and $r \times r$ respectively and $\phi(t)$ is the vector of Legendre scaling function with dimension $N \times 1$, where $N = 2^{K-1}(M + 1)$.

$$\Phi_{\mathrm{nm}}(t) = \left[\, \phi_{10}\ \phi_{11} \dots\ \phi_{1M}\ \phi_{20} \dots \phi_{2M}\ \phi_{2^{K-1}0} \dots\ \phi_{2^{K-1}M}\right]^T$$

And

$$a^i{}_{nm} = \left[a^i_{10}\ a^i_{11} \dots\ a^i_{1M}\ \ a^i_{2^{K-1}0}\ a^i_{2^{K-1}1} \dots\ a^i_{2^{K-1}M}\right] \quad i = 1,2, \dots, s$$

$$b^i{}_{nm} = \left[b^i_{10}\ b^i_{11} \dots\ b^i_{1M}\ \ b^i_{2^{K-1}0}\ b^i_{2^{K-1}1} \dots\ b^i_{2^{K-1}M}\right] \quad i = 1,2, \dots, r$$

3.4 Operational Matrix of Integration (OMI)

Lemma 3.1 *The matrix P is called operational matrix of integration of Legendre scaling function and can be given by*

$$\mathrm{P} = \frac{1}{2^K}\begin{bmatrix} D & U & U & U & \dots & \dots & U \\ O & D & U & U & \dots & \dots & U \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & \dots & O & D & U & U \\ O & O & O & \dots & O & D & U \\ O & O & O & \dots & \dots & O & D \end{bmatrix} \tag{3.7}$$

Where

P $is\ a$ $\left(2^{K-1}(M+1)\right) \times \left(2^{K-1}(M+1)\right)$ operational matrix of integration, $O, U, and\ D$ are $(M+1) \times (M+1)$ matrices and given by

$$U = \begin{bmatrix} 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.8}$$

$$D = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{15}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{15}} & 0 & \frac{1}{\sqrt{35}} & \ddots & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{35}} & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \frac{-1}{\sqrt{4M^2-16M+15}} & 0 & \frac{1}{\sqrt{4M^2-8M+3}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{\sqrt{4M^2-8M+3}} & 0 \end{bmatrix} \tag{3.9}$$

O is matrix that all entire elements are zeros.

To be familiar with operational matrix of integration we will give here the form of matrix for $K = 2, M = 5$ as follows

The D matrix will be of dimension (6×6) and given as in equation (3.11) and the matrix P will be of dimension (12×12) and can be given in compact form as in equation (3.12).

Lemma 3.2 *The integration of multiplication of Legendre scaling function and its transpose in the interval* $t \in [0, 1]$ *is equal to identity matrix since Legendre scaling functions are orthonormals as follows*

$$\int_0^1 \phi(t)\ \phi^{\mathrm{T}}(t)\, dt = \boldsymbol{I_N} \tag{3.10}$$

Where $\boldsymbol{I_N}$, is identity matrix of dimension N, $(N = 2^{K-1}(M+1))$.

$$D = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{15}} & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{15}} & 0 & \frac{1}{\sqrt{35}} & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{35}} & 0 & \frac{1}{\sqrt{63}} & 0 \\ 0 & 0 & 0 & \frac{-1}{\sqrt{63}} & 0 & \frac{1}{\sqrt{99}} \\ 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{99}} & 0 \end{bmatrix} \tag{3.11}$$

$$P = \frac{1}{2^2}\begin{bmatrix} D & U \\ O & D \end{bmatrix} \tag{3.12}$$

3.5 Performance Index Parameterization via Legendre Scaling function

In this section, we would like to construct the formula of performance index to calculate its numerical value easily.

The first step is to integrate the equation (3.2) as follows

$$x(t) - x_o = \int_0^t A\, x(\tau) d\tau + \int_0^t B\, u(\tau) d\tau \tag{3.13}$$

Where x_o is resulted from integration and known as initial condition vector. x_o can be expressed via Legendre scaling function as follows

$$x_o = \frac{\sqrt{2}}{2^{K/2}} \left(\mathbf{I_s} \otimes \phi^T(t)\right)\xi_o \tag{3.14}$$

Where

$$\xi_o = [\alpha_0^1\, \alpha_0^2 \dots \dots\, \alpha_0^s\,] \tag{3.15}$$

$$\alpha_0^i = [x_i(0)\ 0\ 0 \dots 0 | x_i(0)\ 0\ 0 \dots\ 0| \dots | x_i(0)\ 0 \dots 0] \tag{3.16}$$

The second step is to substitute equations (3.5) and (3.6) into equation (3.1) to get

$$J = \int_0^1 \boldsymbol{a^T}\left(\mathbf{I_s} \otimes \phi(t)\right)\boldsymbol{Q}\left(\mathbf{I_s} \otimes \phi^T(t)\right)\boldsymbol{a} + \boldsymbol{b^T}\left(\mathbf{I_r} \otimes \phi(t)\right)\boldsymbol{R}\left(\mathbf{I_r} \otimes \phi^T(t)\right)\boldsymbol{b}\, dt \tag{3.17}$$

By applying Lemma 3.2, equation (3.17) can be simplified as in equation (3.18)

$$J = \boldsymbol{a}^T(\boldsymbol{Q} \otimes \mathbf{I_N})\boldsymbol{a} + \boldsymbol{b}^T(\boldsymbol{R} \otimes \mathbf{I_N})\boldsymbol{b} \tag{3.18}$$

Moreover, equation (3.18) can be written in quadratic form as follows

$$J = [\boldsymbol{a}^T\ \ \boldsymbol{b}^T]\begin{bmatrix} \boldsymbol{Q} \otimes \mathbf{I_N} & \boldsymbol{O}_{Ns\times Nr} \\ \boldsymbol{O}_{Nr\times Ns} & \boldsymbol{R} \otimes \mathbf{I_N} \end{bmatrix}\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} \tag{3.19}$$

3.6 Continuity Test

Wavelets functions are not supported on whole interval $(a \leq t < b)$; so these functions divide the interval of interest to number of sections depending on the value of scaling parameter K; for this reasons we have to add additional constraints to ensure the continuity of the state variables between different sections. There are $(2^{K-1} - 1)$ points at which the continuity of state variables must be tested according to equation (3.20).

$$t_i = \frac{i}{2^{K-1}} \qquad i = 1,2, \dots, 2^{K-1} - 1 \tag{3.20}$$

In addition, there are $(2^{K-1}-1)\,s$ equality constraints can be given as follows

$$(\mathbf{I_s} \otimes \Phi')\boldsymbol{a} = 0_{(2^{K-1}-1)s\times 1} \tag{3.21}$$

However, the matrix of continuity ensured constraints is $(2^{K-1}-1)\times \left(2^{K-1}(M+1)\right)$ and is given by

$$\Phi' = \begin{bmatrix} \phi_{1m}(t_1) & -\phi_{2m}(t_1) & 0 & 0 & 0 & \cdots & 0 \\ 0 & \phi_{2m}(t_2) & -\phi_{3m}(t_2) & 0 & 0 & \dots & 0 \\ 0 & 0 & \phi_{3m}(t_3) & -\phi_{4m}(t_3) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \phi_{(2^{K-1}-1)m}(t_{2^{K-1}-1}) & -\phi_{(2^{K-1}-1)m}(t_{2^{K-1}-1}) \end{bmatrix}$$

(3.22)

3.7 Quadratic Programming Problem

In this section, we try to construct the quadratic form of optimal problem to be easily solved by MATLAB.

Recall equation (3.13), which can be rewritten in the following form

$$[(\mathrm{A} \otimes \mathrm{P^T}) - \mathrm{I}_{Nn} \quad (\mathrm{B} \otimes \mathrm{P^T})] \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = -\xi_o \delta \tag{3.23}$$

Where $\delta = \frac{\sqrt{2}}{2^{K}/_{2}}$

By combining equations (3.21) and (3.23) we get the following form of equality constraints

$$\begin{bmatrix} (\mathrm{A} \otimes \mathrm{P^T}) - \mathrm{I}_{Ns} & (\mathrm{B} \otimes \mathrm{P^T}) \\ (\mathbf{I_s} \otimes \Phi') & 0_{(2^{K-1}-1)s\times Nr} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} -\xi_o \delta \\ 0_{(2^{K-1}-1)s\times 1} \end{bmatrix} \tag{3.24}$$

Equations (3.19) and (3.24) can be rewritten in compact quadratic form as follows

$$\min_{z} \; z^T \boldsymbol{H}\, z \tag{3.25}$$

Subject to equality constraints

$$\boldsymbol{F}\, z = h \tag{3.26}$$

Where

$$z = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}$$

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{Q} \otimes \mathbf{I_N} & \boldsymbol{O}_{Ns\times Nr} \\ \boldsymbol{O}_{Nr\times Ns} & \boldsymbol{R} \otimes \mathbf{I_N} \end{bmatrix}$$

$$F = \begin{bmatrix} (\mathrm{A} \otimes \mathrm{P}^{\mathrm{T}}) - \mathrm{I}_{Ns} & (\mathrm{B} \otimes \mathrm{P}^{\mathrm{T}}) \\ (\mathbf{I_s} \otimes \Phi') & 0_{(2^{K-1}-1)s \times Nr} \end{bmatrix}$$

$$h = \begin{bmatrix} -\xi_o \delta \\ 0_{(2^{K-1}-1)s \times 1} \end{bmatrix}$$

the optimal vector z^* can be also calculated by the standard quadratic programming method [17] as follows

$$z^* = \boldsymbol{H}^{-1} \boldsymbol{F}^T (\boldsymbol{F} \boldsymbol{H}^{-1} \boldsymbol{F}^T)^{-1} h \tag{3.27}$$

3.8 Numerical Examples

3.8.1 Example (1)

Find the optimal control $u^*(t)$ which minimize the following performance index

$$J = \frac{1}{2} \int_0^1 (x^2 + u^2) dt$$

Subject to equality constraints and initial condition as follows

$$\dot{x} = -x + u \quad , x(0)=1$$

To illustrate the proposed method we will solve example (1) in details and step by step as follows

Step (1): define the unknown coefficients of state and control variables according to K and M.

For $K = 2$, and $M = 2$ the unknown coefficients will be as follows

$$\boldsymbol{a} = [a_{10}\ a_{11} a_{12}\ \ a_{20}\ a_{21} a_{22}]$$

$$\boldsymbol{b} = [b_{10}\ b_{11} b_{12}\ \ b_{20}\ b_{21} b_{22}]$$

Step (2): generate the Legendre scaling function vector depending on K and M.

$$\Phi(t) = [\phi_{10}(t)\ \phi_{11}(t)\ \phi_{12}(t)\ \ \phi_{20}(t)\ \phi_{21}(t)\ \phi_{22}(t)\ ]^T$$

Step (3): approximate the state and control variables.

$$x(t) = \sum_{n=1}^{2} \sum_{m=0}^{2} a_{nm}\ \phi_{nm}$$
$$= a_{10}\phi_{10} + a_{11}\phi_{11} + a_{12}\phi_{12} + a_{20}\phi_{20} + a_{21}\phi_{21} + a_{22}\phi_{22}$$

$$u(t) = \sum_{n=1}^{2} \sum_{m=0}^{2} b_{nm}\ \phi_{nm} = b_{10}\phi_{10} + b_{11}\phi_{11} + b_{12}\phi_{12} + b_{20}\phi_{20} + b_{21}\phi_{21} + b_{22}\phi_{22}$$

Step (4): find the vector of initial condition.

$$\delta = \frac{\sqrt{2}}{2^{K}/_{2}} = \frac{1}{\sqrt{2}} \ , \xi_o = [1\ 0\ 0\ \ 1\ 0\ 0\ ]^T$$

Step (5): find the points of ensured continuity and the continuity matrix.

$$t_i = \frac{i}{2^{K-1}} = \frac{1}{2}$$

$$\Phi' = [\phi_{10}(0.5)\ \ \phi_{11}(0.5)\ \ \phi_{12}(0.5)\ \ -\phi_{20}(0.5)\ \ -\phi_{21}(0.5)\ \ -\phi_{22}(0.5)\ ]$$

$$\Phi' = [1.4142\ \ \ 2.4494\ \ \ 3.1622\ \ \ -1.4142\ \ \ 2.4494\ \ \ -3.1622\ ]$$

Step (6): determine the performance index by equation (3.19) as follows.

$$J = 0.5(a_{10}^2 + a_{11}^2 + a_{12}^2 + a_{20}^2 + a_{21}^2 + a_{22}^2 + b_{10}^2 + b_{11}^2 + b_{12}^2 + b_{20}^2 + b_{21}^2 + b_{22}^2)$$

Step (7): solve for optimal vector z^* using equation (3.27) or quadratic programming method in MATLAB.

$$z^* = \begin{bmatrix} 0.5122, -0.1001, 0.0095, 0.2684, -0.0464, 0.0050, \dots \\ -0.1829, 0.0465, 0.0060, -0.0518, 0.0304, 0.0084 \end{bmatrix}^T$$

Step (8): substitute the optimal values from step (7) into step (6), and obtain the optimal value $J = 0.192998105237$.

We solve the same problem using different values of K and M and the optimal values are as shown in Table (3.1).

Table (3.1): Optimal Values for Example (1)

M	K	J	Deviation Error
3	**2**	0.192909334109	3.6×10^{-8}
4	**2**	0.192909298711	6.1×10^{-10}
5	**2**	0.192909298093	6.7×10^{-12}

The exact optimal value of this problem as in [13] is 0.1929092981. The optimal state and control trajectories for $M = 3$ and $M = 5$ are shown in Figures (3.1) and (3.2) respectively. Since our system is linear; so the state and control trajectories should approach zero after finite time. On the other hand, we focus on the interval $[0, 1]$ to compare our plots with other researchers to show effectiveness of the proposed method. Moreover, increasing the order M will enhance the accuracy of results.

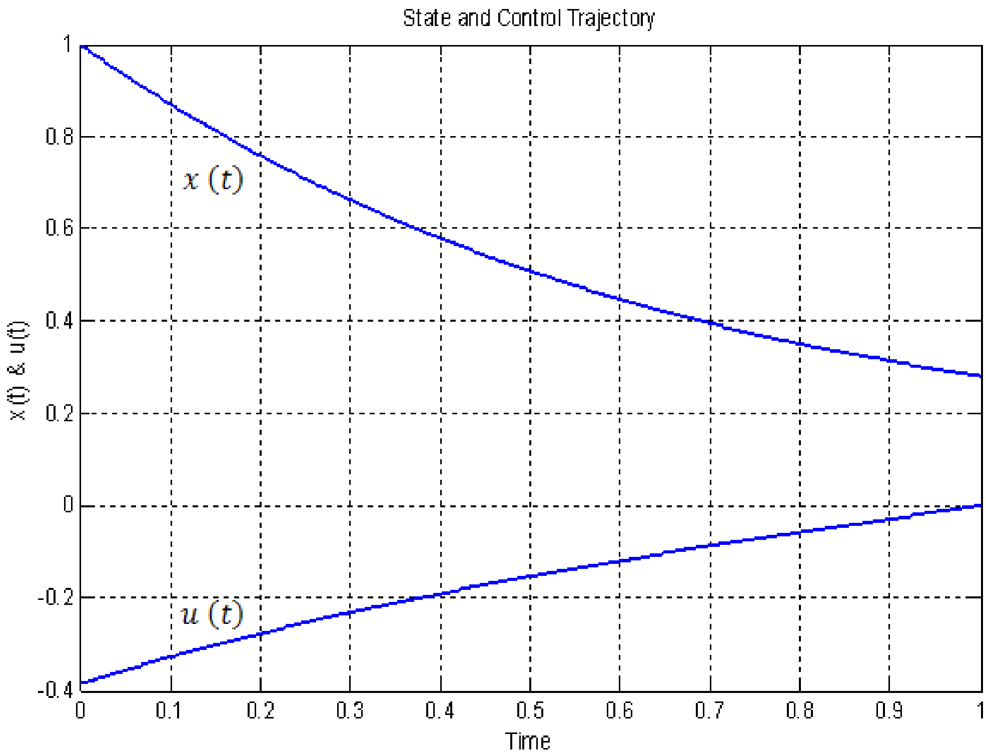

Figure (3.1): Optimal State and Control Trajectories for Example (1) at M =3

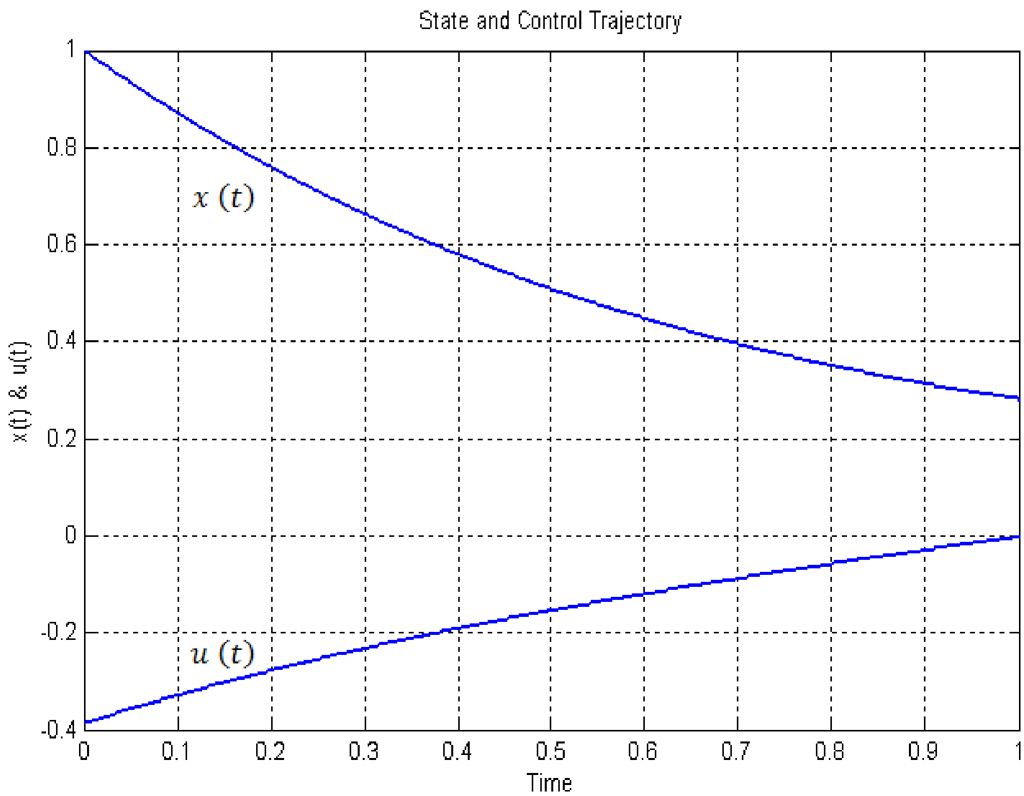

Figure (3.2): Optimal State and Control Trajectories for Example (1) at M =5

3.8.2 Example (2)

Find the optimal control $u^*(t)$ which minimize the following performance index

$$J = \int_0^1 (x_1^2 + x_2^2 + 0.005u^2)dt$$

Subject to equality constraints and initial conditions as follows

$$\dot{x}_1 = x_2 \qquad , x_1(0) = 0$$

$$\dot{x}_2 = -x_2 + u \quad , x_2(0) = -1$$

Using the steps of example (1) to solve this problem for $K = 2$, $M = 4, 5,$ and 6 we obtain the optimal values as shown in Table (3.2).

Table (3.2): Optimal Values for Example (2)

M	K	J	Deviation Error
4	**2**	0.07112166296	1.76×10^{-3}
5	**2**	0.069603214383	2.42×10^{-4}
6	**2**	0.069365200694	4.26×10^{-6}

To show effectiveness and superiority of our proposed method, we compare our results with other methods as shown in Table (3.3).

Table (3.3): Comparison between Optimal Values for Example (2)

Source	Used Method	J	Deviation Error
Exact Value	---	0.06936094	0
Hsieh [18]	Space Function	0.0702	8.4×10^{-4}
Neuman & Sen [19]	Cubic Splines	0.06989	5.3×10^{-4}
Vlassenbroeck[20]	Chebyshev Poly.	0.069368	7.1×10^{-6}
Jaddu [1]	Chebyshev Poly.	0.0693689	7.96×10^{-6}
Majdalawi [15]	Legendre Poly.	0.0693688962	7.956×10^{-6}
This Research	LSF	0.069365200694	4.26×10^{-6}

The optimal state and control trajectories for $K = 2, M = 6$ are shown in Figures (3.3) and (3.4) respectively. These figures are exactly as obtained in [1]. Moreover, $x_2(t)$ converges rapidly to zero after 0.8 second and $u(t)$ after 0.35 second. On the other hand, $x_1(t)$ settles to zero as t goes to ∞.

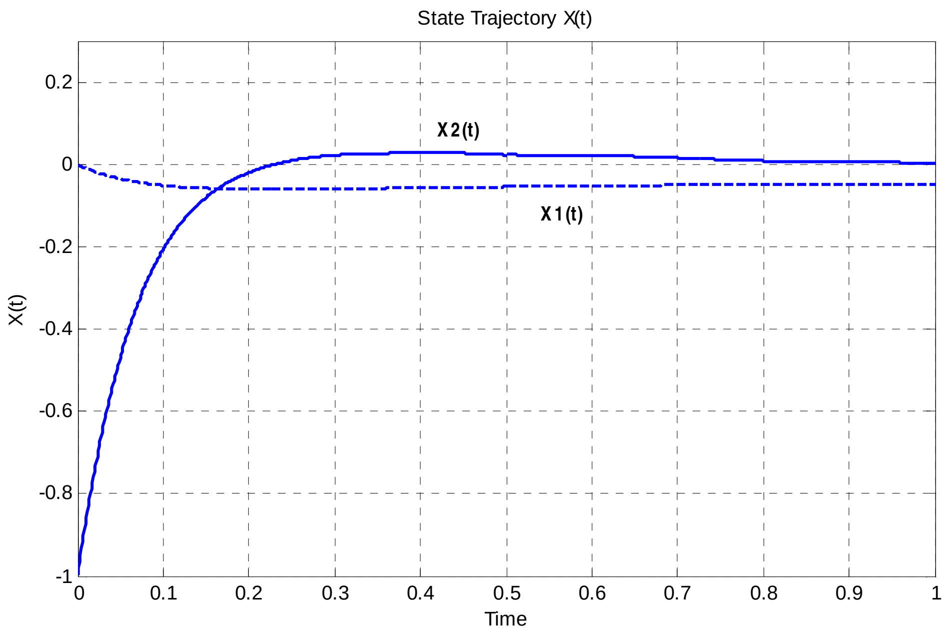

Figure (3.3): Optimal State Trajectories for Example (2) for M =6

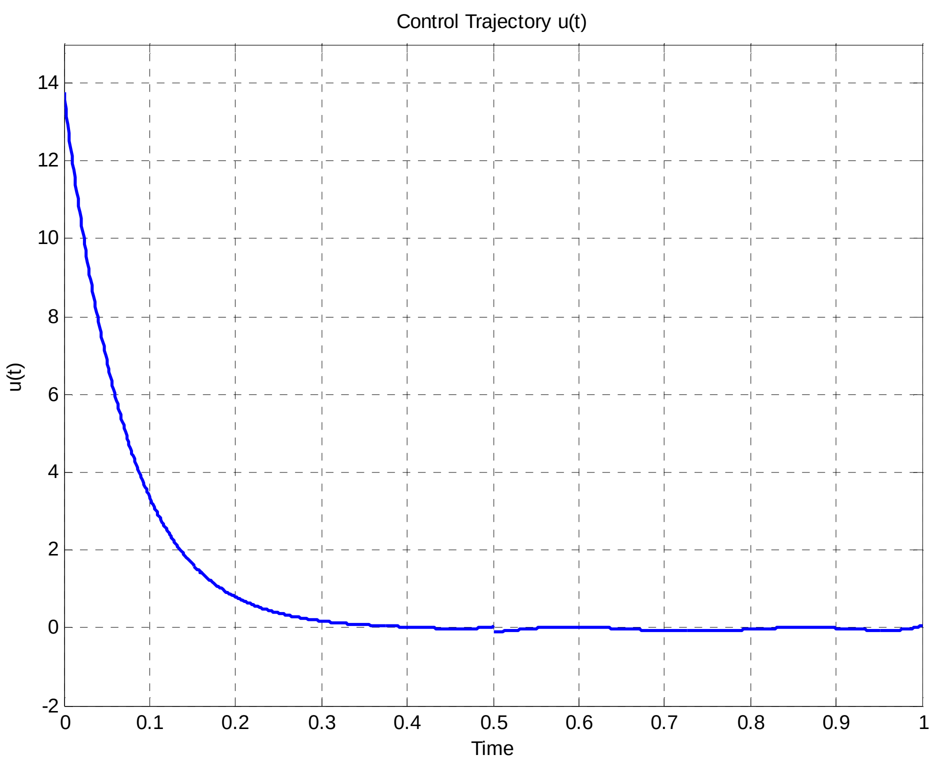

Figure (3.4): Optimal Control Trajectory for Example (2) for M =6

Chapter Four

Linear Time Varying Quadratic Optimal Control Problem

4.1 Introduction

There are many developed techniques for dealing with time varying OCP_S based on Fourier series[21], Legendre polynomials[15], Chebyshev polynomials[1], Chebyshev wavelets[12], and Legendre wavelets[8].

In this chapter, we will expand the developed technique in Chapter Three to solve time varying OCP_S. Moreover, the property of multiplication of two Legendre scaling functions will be presented. In addition, this method will be the basic for next chapter for solving nonlinear OCP_S .

4.2 Multiplication of Two Legendre Scaling Functions

In this section, we will introduce the property of multiplication of two Legendre scaling function; since this step is important to deal with time varying and nonlinear OCP_S.

***Lemma* 4.1:** *The multiplication of two Legendre scaling functions will be zero if two functions are in different sections.*

***Theorem* 4.1:** *Given two Legendre scaling functions $\phi_{nm}(t), \phi_{ns}(t)$ and assuming $m \leq s$, then the multiplication of these two scaling function is given by following formula*

$$\phi_{nm}(t)\,\phi_{ns}(t) = \frac{2^{K/2}}{2}\sqrt{(2m+1)(2s+1)}\sum_{j=0}^{m}\frac{\gamma_j}{\sqrt{\dfrac{2(m+s-2j)+1}{2}}}\,\phi_{n(m+s-2j)}(t) \tag{4.1}$$

Where

$$a_j = \frac{(2j-1)!!}{j!}$$

$$\gamma_j = \frac{a_{m-j}\,a_j\,a_{s-j}}{a_{m+s-j}}\left(\frac{2m+2s-4j+1}{2m+2s-2j+1}\right)$$

Note that j !! is the double factorial of j and $-1!! = 1$ as special case.

Proof of equation (4.1) and many examples of multiplication can be found in details in [14].

4.3 Time Varying Linear Optimal Control Problem

In this section, we present the form of performance index, which is remained as in equation (3.1) and the form of state equations which becomes linear time-varying. Moreover, the optimal control problem will be as follows

Find the optimal control $u^*(t)$ that minimize the quadratic performance index

$$J = \int_0^1 (\boldsymbol{x}^T \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{R}\boldsymbol{u})\, dt \tag{4.2}$$

Subject to time varying state equations and initial condition vector

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\, \boldsymbol{x}(t) + \boldsymbol{B}(t)\, \boldsymbol{u}(t) \quad , \boldsymbol{x}(0) = \boldsymbol{x_o} \tag{4.3}$$

Where $t \in [0, 1]$, $x\, , x_o \in R^n$, $u \in R^m$, $A(t), B(t)$ are $n \times n$ and $n \times m$ time-varying matrices respectively. $\boldsymbol{Q}\ is\ an\ n \times n$ positive semidefinite matrix and $\boldsymbol{R}\ is\ an\ m \times m$ positive definite matrix.

4.3.1 Parameterization via Legendre Scaling Function

State and control variables can be expanded as the same manner as in Chapter three via Legendre scaling function as follows

$$x_i(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M} a_{nm}^i\, \phi_{nm}(t) = \Phi^T(t)\, \boldsymbol{a} \quad i = 1,2, \dots, s \tag{4.4}$$

$$u_i(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M} b_{nm}^i\, \phi_{nm}(t) = \Phi^T(t)\, \boldsymbol{b} \quad i = 1,2, \dots, r \tag{4.5}$$

These equations can be written in compact form as follows

$$\boldsymbol{x}(t) = \left(\mathbf{I_s} \otimes \phi^T(t)\right)\boldsymbol{a} \tag{4.6}$$

$$\boldsymbol{u}(t) = \left(\mathbf{I_r} \otimes \phi^T(t)\right)\boldsymbol{b} \tag{4.7}$$

Where $\mathbf{I_s}, \mathbf{I_r}$ are identity matrices of dimension $s \times s$ and $r \times r$ respectively and $\phi(t)$ is the vector of Legendre scaling function with dimension $N \times 1$, where $N = 2^{K-1}(M+1)$.

$$\Phi_{\mathrm{nm}}(t) = \left[\, \phi_{10}\, \phi_{11} \dots \phi_{1M}\, \phi_{20} \dots \phi_{2M}\, \phi_{2^{K-1}0} \dots \phi_{2^{K-1}M}\right]^T$$

And

$$\boldsymbol{a} = \left[a_{10}^i \dots a_{10}^s\ \ a_{11}^1 \dots a_{11}^s\, a_{2^{K-1}M}^1 \dots a_{2^{K-1}M}^s\right]$$

$$\boldsymbol{b} = \left[b_{10}^1 \dots b_{10}^r\ \ b_{11}^1 \dots b_{11}^r\, b_{2^{K-1}M}^1 \dots b_{2^{K-1}M}^r\right]$$

We should note here that the unknown vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are slightly different than in Chapter three.

4.3.2 Parameterization $A(t)$ and $B(t)$ via Legendre Scaling Function

The time varying matrices $A(t)$ and $B(t)$ can be expressed in terms of Legendre scaling function as follows

$$\mathbf{A}(t) = \sum_{i=1}^{2^{K-1}} \sum_{j=0}^{M} \mathbf{A}_{ij}\, \phi_{ij}(t) \tag{4.8}$$

Equation (4.8) can be written in vector form as follows.

$$\mathbf{A}(t) = \left[\mathbf{A}_{10}\ \mathbf{A}_{11} \ldots\ \mathbf{A}_{1M}\ \mathbf{A}_{2^{K-1}0} \ldots \mathbf{A}_{2^{K-1}M}\right] \Phi(t) \tag{4.9}$$

Where $\mathbf{A}_{ij}$ is $s \times s$ constant matrix of the coefficients of Legendre scaling function $\phi_{ij}(t)$. These matrices can be found by the following formula

$$\mathbf{A}_{ij} = \int_{\frac{2i-2}{2^K}}^{\frac{2i}{2^K}} \mathbf{A}(t)\, \phi_{ij}(t)\, dt \tag{4.10}$$

In similar manner, $B(t)$ can be expanded in terms of Legendre scaling function as follows

$$\mathbf{B}(t) = \sum_{i=1}^{2^{K-1}} \sum_{j=0}^{M} \mathbf{B}_{ij}\, \phi_{ij}(t) \tag{4.11}$$

Also equation (4.11) can be written in vector form as follows

$$\mathbf{B}(t) = \left[\mathbf{B}_{10}\ \mathbf{B}_{11} \ldots\ \mathbf{B}_{1M}\ \mathbf{B}_{2^{K-1}0} \ldots \mathbf{B}_{2^{K-1}M}\right] \Phi(t) \tag{4.12}$$

Where $\mathbf{B}_{ij}$ is $s \times r$ constant matrix of the coefficients of Legendre scaling function $\phi_{ij}(t)$. These matrices can be found by the following formula

$$\mathbf{B}_{ij} = \int_{\frac{2i-2}{2^K}}^{\frac{2i}{2^K}} \mathbf{B}(t)\, \phi_{ij}(t)\, dt \tag{4.13}$$

4.3.3 Initial Conditions parameterization via Legendre Scaling Function

Initial condition can be expanded in terms of Legendre scaling function as follows

$$\boldsymbol{x}_o = \delta\, \Phi^{\mathrm{T}}\, \alpha^0 \tag{4.14}$$

Where

$$\delta = \frac{\sqrt{2}}{2^{K/2}}$$

$$\alpha^0 = \left[\alpha_{10}^0\ 0 \dots 0\ \alpha_{20}^0\ 0 \dots 0\ \alpha_{2^{K-1}0}^0\ 0 \dots 0\right] \text{ and } \alpha_{10}^0 = [x_1(0)\ x_2(0) \dots\ x_s(0)]$$

Here we should note that the vector α^0 differs from the vector ξ_o in equation (3.15).

4.3.4 Performance Index parameterization via Legendre Scaling Function

To approximate the performance index of time varying system, we substitute equations (4.6) and (4.7) into equation (4.2) to obtain

$$J = \int_0^1 \boldsymbol{a}^{\boldsymbol{T}}(\phi(t)\ \otimes \mathbf{I_s})\boldsymbol{Q}(\phi^T(t)\ \otimes \mathbf{I_s})\boldsymbol{a} + \boldsymbol{b}^{\boldsymbol{T}}(\phi(t)\ \otimes \mathbf{I_r})\boldsymbol{R}(\phi^T(t)\ \otimes \mathbf{I_r})\boldsymbol{b}\, dt \qquad (4.15)$$

By applying Lemma 3.2, equation (4.15) can be simplified as follows

$$J = \boldsymbol{a}^T(\ \mathbf{I_N} \otimes\ \boldsymbol{Q})\boldsymbol{a} + \boldsymbol{b}^T(\ \mathbf{I_N} \otimes \boldsymbol{R})\boldsymbol{b} \qquad (4.16)$$

Moreover, equation (4.16) can be written in quadratic form as follows

$$J = [\boldsymbol{a}^T\ \ \boldsymbol{b}^T]\begin{bmatrix}\mathbf{I_N} \otimes \boldsymbol{Q} & \boldsymbol{O}_{Ns\times Nr} \\ \boldsymbol{O}_{Nr\times Ns} & \mathbf{I_N} \otimes \boldsymbol{R}\end{bmatrix}\begin{bmatrix}\boldsymbol{a} \\ \boldsymbol{b}\end{bmatrix} \qquad (4.17)$$

4.3.5 State Equations parameterization via Legendre Scaling Function

To approximate state equations in terms of unknown coefficients of state and control variables, we integrate equation (4.3) as follows

$$\boldsymbol{x}(t) - \boldsymbol{x}_o = \int_0^t \mathbf{A}(t)\ \boldsymbol{x}(\tau)\, d\tau + \int_0^t \mathbf{B}(t)\ \boldsymbol{u}(\tau)\, d\tau \qquad (4.18)$$

By substituting equations (4.4), (4.5), (4.9), (4.12) and (4.14) into equation (4.18), we obtain

$$\Phi^T(t)\ \boldsymbol{a} - \Phi^{\mathrm{T}}(t)\delta\ \alpha^0 =$$

$$\int_0^t [\mathbf{A}_{10} \dots \mathbf{A}_{2^{K-1}M}]\ \Phi(t)\ \Phi^{\mathrm{T}}(t)\boldsymbol{a}\ + \int_0^t [\mathbf{B}_{10} \dots \mathbf{B}_{2^{K-1}M}]\ \Phi(t)\ \Phi^{\mathrm{T}}(t)\boldsymbol{b}\, dt \qquad (4.19)$$

By using the results of theorem (4.1) which can be found in details in [14],we get

$$\mathbf{V}^T\ \Phi(t)\ \Phi^{\mathrm{T}}(t) = \Phi^{\mathrm{T}}(t)\ \widetilde{\mathbf{V}} \qquad (4.20)$$

Where $\mathbf{V}^T = \left[v_{10}\ \dots v_{1M}\ v_{2^{K-1}0}\ \dots\ v_{2^{K-1}M}\ \right]$ and $\widetilde{\mathbf{V}}$ is a $\left(2^{k-1}(M+1)\right) \times \left(2^{k-1}(M+1)\right)$ product operational matrix (POM). To illustrate the calculation procedures, we choose $M = 2$ and $K = 2$ Thus, we have

$$\mathbf{V} = [v_{10}\, v_{11}\ \ v_{12}\, v_{20}\, v_{21}\ \ v_{22}\ \ ]^T \qquad (4.21)$$

$$\Phi(t) = [\ \phi_{10}\ \phi_{11}\ \phi_{12}\ \phi_{20}\ \phi_{21}\phi_{22}\ ]^T \qquad (4.22)$$

Multiply equation (4.22) in its transpose, we get

$$\Phi(t)\Phi^{\mathrm{T}}(t)=\begin{bmatrix}\phi_{10}\phi_{10} & \phi_{10}\phi_{11} & \phi_{10}\phi_{12} & \phi_{10}\phi_{20} & \phi_{10}\phi_{21} & \phi_{10}\phi_{22}\\ \phi_{11}\phi_{10} & \phi_{11}\phi_{11} & \phi_{11}\phi_{12} & \phi_{10}\phi_{20} & \phi_{11}\phi_{21} & \phi_{11}\phi_{22}\\ \phi_{12}\phi_{10} & \phi_{12}\phi_{11} & \phi_{12}\phi_{12} & \phi_{10}\phi_{20} & \phi_{12}\phi_{21} & \phi_{12}\phi_{22}\\ \phi_{20}\phi_{10} & \phi_{20}\phi_{11} & \phi_{20}\phi_{12} & \phi_{10}\phi_{20} & \phi_{20}\phi_{21} & \phi_{20}\phi_{22}\\ \phi_{21}\phi_{10} & \phi_{21}\phi_{11} & \phi_{21}\phi_{12} & \phi_{10}\phi_{20} & \phi_{21}\phi_{21} & \phi_{21}\phi_{22}\\ \phi_{22}\phi_{10} & \phi_{22}\phi_{11} & \phi_{22}\phi_{12} & \phi_{10}\phi_{20} & \phi_{22}\phi_{21} & \phi_{22}\phi_{22}\end{bmatrix} \tag{4.23}$$

Applying Lemma (4.1) on equation (4.23), we get

$$\Phi(t)\Phi^{\mathrm{T}}(t)=\begin{bmatrix}\sqrt{2}\,\phi_{10} & \sqrt{2}\phi_{11} & \sqrt{2}\phi_{12} & 0 & 0 & 0\\ \sqrt{2}\phi_{11} & \sqrt{2}\phi_{10}+\frac{4}{\sqrt{10}}\phi_{12} & \frac{4}{\sqrt{10}}\phi_{11} & 0 & 0 & 0\\ \sqrt{2}\phi_{12} & \frac{4}{\sqrt{10}}\phi_{11} & \sqrt{2}\phi_{10}+\frac{20}{7\sqrt{10}}\phi_{12} & 0 & 0 & 0\\ 0 & 0 & 0 & \sqrt{2}\,\phi_{20} & \sqrt{2}\phi_{12} & \sqrt{2}\phi_{22}\\ 0 & 0 & 0 & \sqrt{2}\phi_{21} & \sqrt{2}\phi_{20}+\frac{4}{\sqrt{10}}\phi_{22} & \frac{4}{\sqrt{10}}\phi_{21}\\ 0 & 0 & 0 & \sqrt{2}\phi_{22} & \frac{4}{\sqrt{10}}\phi_{21} & \sqrt{2}\phi_{20}+\frac{20}{7\sqrt{10}}\phi_{22}\end{bmatrix} \tag{4.24}$$

By using equation (4.21), $\widetilde{\mathbf{V}}$ can be given in the following form

$$\widetilde{\mathbf{V}}=\begin{bmatrix}V_1 & 0 & \cdots & 0\\ 0 & V_2 & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ 0 & \dots & 0 & V_{2^{K-1}}\end{bmatrix} \tag{4.25}$$

Where V_i for $K=2, M=4$ is given in [14] by the following matrix

$$V_i=\sqrt{2}\begin{bmatrix}v_{i0} & v_{i1} & v_{i2} & v_{i3} & v_{i4}\\ v_{i1} & v_{i0}+\frac{4}{\sqrt{20}}v_{i2} & \frac{4}{\sqrt{20}}v_{i1}+\frac{3\sqrt{3}}{35}v_{i3} & \frac{3\sqrt{3}}{35}v_{i2}+\frac{4}{\sqrt{21}}v_{i4} & \frac{4}{\sqrt{21}}v_{i3}\\ v_{i2} & \frac{4}{\sqrt{20}}v_{i1}+\frac{3\sqrt{3}}{35}v_{i3} & v_{i0}+\frac{\sqrt{20}}{7}v_{i2}+\frac{6}{7}v_{i4} & \frac{3\sqrt{3}}{35}v_{i1}+\frac{4\sqrt{5}}{\sqrt{15}}v_{i3} & \frac{6}{7}v_{i2}+\frac{20\sqrt{5}}{77}v_{i4}\\ v_{i3} & \frac{3\sqrt{3}}{35}v_{i2}+\frac{4}{\sqrt{21}}v_{i4} & \frac{3\sqrt{3}}{35}v_{i1}+\frac{4\sqrt{5}}{\sqrt{15}}v_{i3} & \frac{6}{7}v_{i2}+\frac{20\sqrt{5}}{77}v_{i4} & \frac{4}{\sqrt{21}}v_{i1}+\frac{6}{11}v_{i3}\\ v_{i4} & \frac{4}{\sqrt{21}}v_{i3} & \frac{6}{7}v_{i2}+\frac{20\sqrt{5}}{77}v_{i4} & \frac{4}{\sqrt{21}}v_{i1}+\frac{6}{11}v_{i3} & v_{i0}+\frac{100}{77\sqrt{5}}v_{i2}+\frac{489}{1001}v_{i4}\end{bmatrix} \tag{4.26}$$

According to the results obtained in equation (4.20), we have

$$[\mathbf{A}_{10}\dots\mathbf{A}_{2^{K-1}M}]\Phi(t)\Phi^{\mathrm{T}}(t)=\Phi^{\mathrm{T}}(t)\,\tilde{A} \tag{4.27}$$

$$[\mathbf{B}_{10}\dots\mathbf{B}_{2^{K-1}M}]\Phi(t)\Phi^{\mathrm{T}}(t)=\Phi^{\mathrm{T}}(t)\,\tilde{B} \tag{4.28}$$

Where $\tilde{A}$ and $\tilde{B}$ are constant matrices of dimension $sN\times sN$ and $sN\times rN$ respectively.

Now equation (4.19) can be simplified using equations (4.27) and (4.28) as follows

$$\Phi^T(t)\,\boldsymbol{a} - \Phi^{\mathrm{T}}(t)\delta\,\alpha^0 = \int_0^t \Phi^{\mathrm{T}}(t)\,\tilde{A}\,\boldsymbol{a}\,dt \;+ \int_0^t \Phi^{\mathrm{T}}(t)\,\tilde{B}\,\boldsymbol{b}\,dt \qquad (4.29)$$

To eliminate the integral operator, we apply Lemma (3.1) on equation (4.29), we get

$$\Phi^T(t)\,\boldsymbol{a} - \Phi^{\mathrm{T}}(t)\delta\,\alpha^0 = \Phi^{\mathrm{T}}(t)\,\mathbf{P}^T\,\tilde{A}\,\boldsymbol{a} + \Phi^{\mathrm{T}}(t)\,\mathbf{P}^T\,\tilde{B}\,\boldsymbol{b} \qquad (4.30)$$

The multiplication in equation (4.30) is block-wise, to change into element-wise multiplication, we can use Kronecker product [22], we get

$$(\Phi^T(t) \otimes \mathbf{I_s})\boldsymbol{a} - (\Phi^T(t) \otimes \mathbf{I_s})\delta\,\alpha^0 =$$

$$(\Phi^{\mathrm{T}}(t)\,\mathbf{P}^T \otimes \mathbf{I_s})\,\tilde{A}\,\boldsymbol{a} + (\Phi^{\mathrm{T}}(t)\,\mathbf{P}^T \otimes \mathbf{I_s})\,\tilde{B}\,\boldsymbol{b} \qquad (4.31)$$

4.4 Quadratic Programming Problem

In this section, as in section (3.7), we construct the quadratic form of optimal problem to be easily solved by MATLAB.

The compact quadratic form can be rewritten as follows

$$\min_{z} \; z^T\,\boldsymbol{H}\,z \qquad (4.28)$$

Subject to equality constraints

$$\boldsymbol{F}\,z = h \qquad (4.29)$$

Where

$$z = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}$$

$$\boldsymbol{H} = \begin{bmatrix} \mathbf{I_N} \otimes \boldsymbol{Q} & \boldsymbol{O}_{Ns\times Nr} \\ \boldsymbol{O}_{Nr\times Ns} & \mathbf{I_N} \otimes \boldsymbol{R} \end{bmatrix}$$

$$\boldsymbol{F} = \begin{bmatrix} (\mathrm{P}^{\mathrm{T}} \otimes \mathbf{I_s})\tilde{A} - \mathrm{I}_{Ns} & (\,\mathbf{P}^T \otimes \mathbf{I_s})\,\tilde{B} \\ (\Phi' \otimes \mathbf{I_s}) & 0_{(2^{K-1}-1)s\times Nr} \end{bmatrix}$$

$$h = \begin{bmatrix} -\alpha^o\delta \\ 0_{(2^{K-1}-1)s\times 1} \end{bmatrix}$$

4.5 Numerical Examples

Find the optimal control $u^*(t)$ which minimize the following performance index

$$J = \frac{1}{2}\int_0^1 (x^2 + u^2)dt$$

Subject to equality constraints and initial condition as follows

$$\dot{x} = tx + u \quad ,x(0)=1$$

To illustrate the proposed method we will solve this example in details and step by step as follows

Step (1): define the unknown coefficients of state and control variables according to $Kand\ M$. For $K = 2$, and $M = 2$ the unknown coefficients will be as follows

$$\boldsymbol{a} = [a_{10}\ a_{11}\ a_{12}\ a_{20}\ a_{21}\ a_{22}\ ]$$

$$\boldsymbol{b} = [b_{10}\ b_{11}\ b_{12}\ b_{20}\ b_{21}\ b_{22}]$$

Step (2): generate the Legendre scaling function vector depending on $Kand\ M$.

$$\Phi(t) = [\phi_{10}(t)\ \phi_{11}(t)\ \phi_{12}(t)\ \ \phi_{20}(t)\ \phi_{21}(t)\ \phi_{22}(t)\ ]^T$$

Step (3): approximate the state and control variables.

$$x(t) = \sum_{n=1}^{2}\sum_{m=0}^{2} a_{nm}\ \phi_{nm}$$

$$u(t) = \sum_{n=1}^{2}\sum_{m=0}^{2} b_{nm}\ \phi_{nm}$$

Step (4): find the vector of initial condition.

$$\delta = \frac{\sqrt{2}}{2^{K/2}} = \frac{1}{\sqrt{2}}\ , \alpha_o = [1\ 0\ 0\ \ 1\ 0\ 0]^T$$

Step (5): find the points of ensured continuity and the continuity matrix.

$$t_i = \frac{i}{2^{K-1}} = \frac{1}{2}$$

$$\Phi' = [\phi_{10}(0.5)\ \ \phi_{11}(0.5)\ \ \phi_{12}(0.5)\ \ -\phi_{20}(0.5)\ \ -\phi_{21}(0.5)\ \ -\phi_{22}(0.5)\ \ ]$$

Step (6): find $\tilde{A}$ by approximating $\mathbf{A}(t)$ using equation (4.10).

Now solving the quadratic programming problem, we obtained $J = 0.484290813333$

To show the convergence speed to optimal value, we solve the same problem for different values of K, and M, then the optimal values are recorded in Table (4.1).

To show effectiveness of our proposed method we compare our results with other methods as shown in Table (4.2).

Table (4.1): Optimal values for different Values of M and K

Approx. Order	J
$\boldsymbol{M = 2, K = 2}$	0.484290813333
$\boldsymbol{M = 3, K = 2}$	0.484267886132
$\boldsymbol{M = 4, K = 2}$	0.484267700376

Table (4.2): Comparison between Optimal Values of Example

Source	Used Method	J	Approx. Order
Elnagar [23]	Cell Averaging Spectral Chebyshev	0.48427022	$N = 4$
Elnagar [23]	Cell Averaging Spectral Chebyshev	0.48426764	$N = 6$
This Research	LSF	0.48426770037	$M = 4, K = 2$

The optimal state and control trajectories for $K = 2, M = 2$ and for $K = 2, M = 4$ are shown in Figures (4.1) and (4.2) respectively. Moreover, we notice from Figures that we can get more accurate results by increasing M.

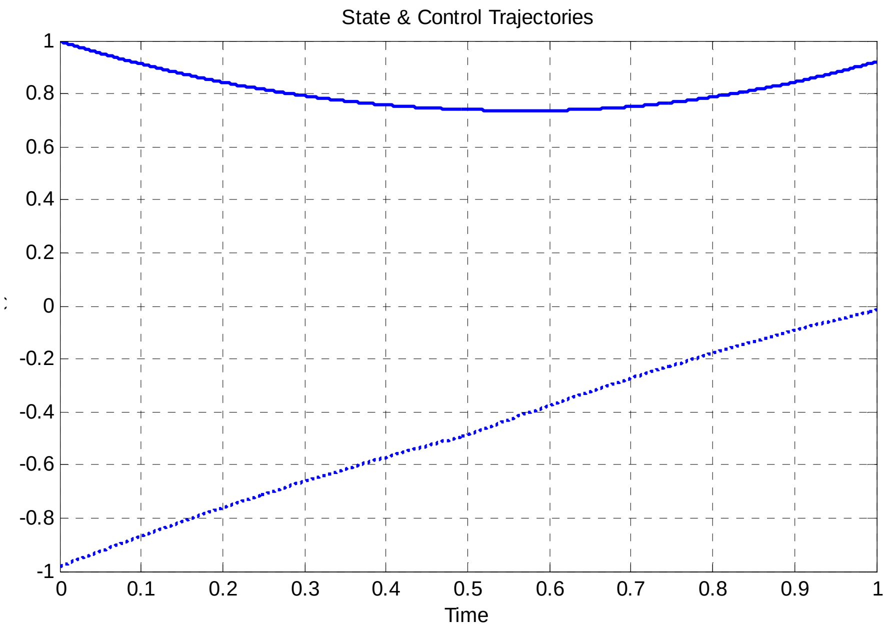

Figure (4.1): Optimal State and Control Trajectories at $M = 2$

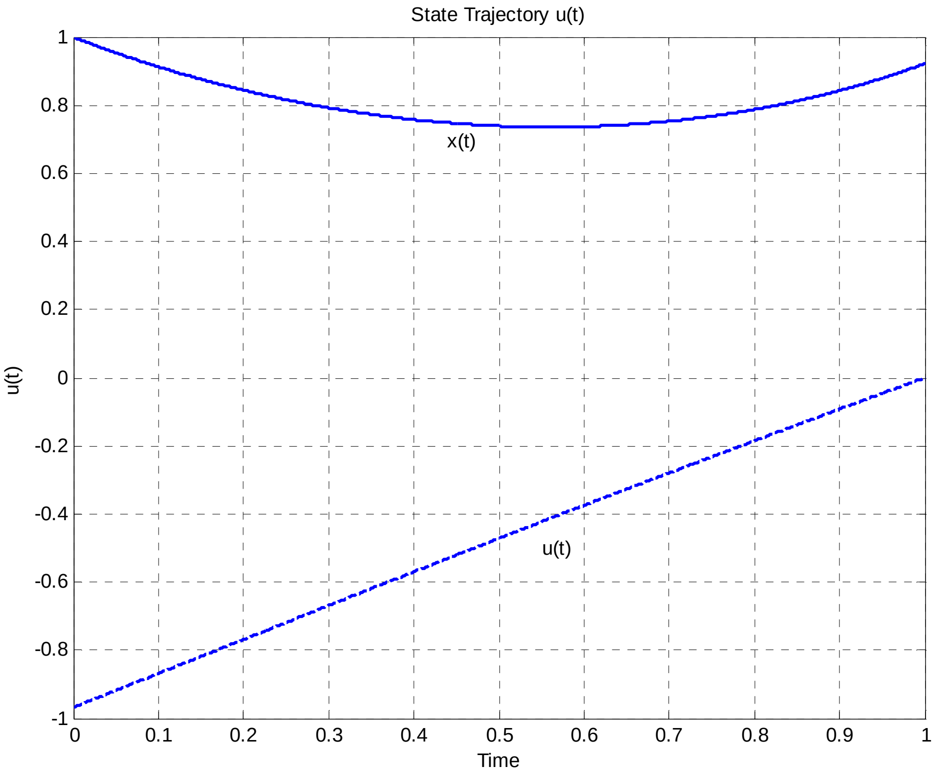

Figure (4.2): Optimal State and Control Trajectories at $M = 4$

Chapter Five

Nonlinear Quadratic Optimal Control Problem

5.1 Introduction

In this chapter, nonlinear OCP_S will be solved depending on the proposed method in Chapter Four. In addition, iterative technique will be used to convert the complex nonlinear OCP_S into sequence of linear time varying OCP_S[5-7] which are much easier to solve by any software package such as MATLAB.

Many researchers proposed different methods for solving nonlinear OCP_S. For example, Razzaghi and Yousefi [8] used Legendre wavelets and Gauss node method to solve nonlinear OCP_S. Tomas and et al.[24] presented a parametric approach based on reducing the nonlinear OCP to a sequence of linear time varying ones. Zheng and Yang[25] proposed method for solving nonlinear differential equations depending on Legendre wavelets and neural network. Jaddu[26] proposed direct solution of nonlinear OCP_S using state parameterization based on Chebyshev polynomials and the second method of the quasilinearzation. By which, difficult nonlinear OCP_S are converted directly into a sequence of quadratic programming problems.

In this thesis, the nonlinear OCP_S will be replaced by a sequence of time varying OCP_S based on the iterative technique. Then, the proposed method in Chapter Four will be applied.

5.2 Problem Statement of Nonlinear OCP

The optimal control problem can be considered as finding the optimal controller $u^*(t)$ that minimize the performance index

$$J = \int_0^{t_f} (x^T Q x + u^T R u)\, dt \tag{5.1}$$

Subject to nonlinear constraints and initial condition

$$\dot{x} = f(x(t), u(t), t) = A(x)\, x(t) + B(x)\, u(t)\, , x(0) = x_o \tag{5.2}$$

Where $t \in [0, t_f]$, $x, x_o \in R^n$, $u \in R^m$, A, B are $n \times n$ and $n \times m$ constant matrices respectively. Q is an $n \times n$ positive semidefinite matrix and R is an $m \times m$ positive definite matrix and f is assumed to be continuous differential function with respect to all its arguments.

5.3 Iterative Technique

In this section, we will introduce a modern technique for handling nonlinear dynamical systems in which the original nonlinear OCP_S are replaced by a sequence of linear time varying OCP_S under Lipschitz condition[27]. Mathematically if $A(x)$ is locally Lipschitz, then the nonlinear equation (5.2) can be replaced by the following linear time varying system.

$$\dot{x}^{[0]} = A(x_0)x^{[0]} + B(x_0)\, u^{[0]} \quad , \; x^{[0]}(0) = x_0 \tag{5.3}$$

and for $k \geq 1$

$$\dot{x}^{[k]} = A\big(x^{[k-1]}(t)\,\big)x^{[k]} + B\left(x^{[k-1]}(t)\right) u^{[k]} \quad , \; x^{[k]}(0) = x_0 \tag{5.4}$$

***Theorem* 5.1:** *Suppose that the nonlinear equation (5.2) has a unique solution on the interval* $[0, t]$ *denoted by* $x(t)$ *and assume that* $A(x): R^n \to R^n$ *is locally Lipschitz. Then the sequence of functions defined in (5.3)-(5.4) converge uniformly on* $[0, t]$ *to the solution* $x(t)$.

Iterative technique have some advantages that makes it an attractive tool for solving any nonlinear equation, that satisfies the local Lipschitz condition. (1) Iterative technique provides an accurate representation of the nonlinear system after small number of iterations. (2) Common linear techniques can be applied on nonlinear systems by using iterative technique.

Now OCP described in (5.1)-(5.2) can be replaced by the following representation based on iterative technique

Minimize

$$J^{[0]} = \int_0^{t_f} (x^{[0]^T} Q\, x^{[0]} + u^{[0]^T} R\, u^{[0]})\, dt \tag{5.5}$$

Subject to

$$\dot{x}^{[0]} = A(x_0)x^{[0]} + B(x_0)\, u^{[0]} \quad , \qquad x^{[0]}(0) = x_0 \tag{5.6}$$

And for k ≥1

Minimize

$$J^{[k]} = \int_0^{t_f} (x^{[k]^T} Q\, x^{[k]} + u^{[k]^T} R\, u^{[k]})\, dt \tag{5.7}$$

Subject to

$$\dot{x}^{[k]} = A\big(x^{[k-1]}(t)\,\big)x^{[k]} + B\left(x^{[k-1]}(t)\right) u^{[k]} \quad , \; x^{[k]}(0) = x_0 \tag{5.8}$$

The procedures required to solve nonlinear quadratic OCP_S using iterative technique can be summarized as in Figure (5.1).

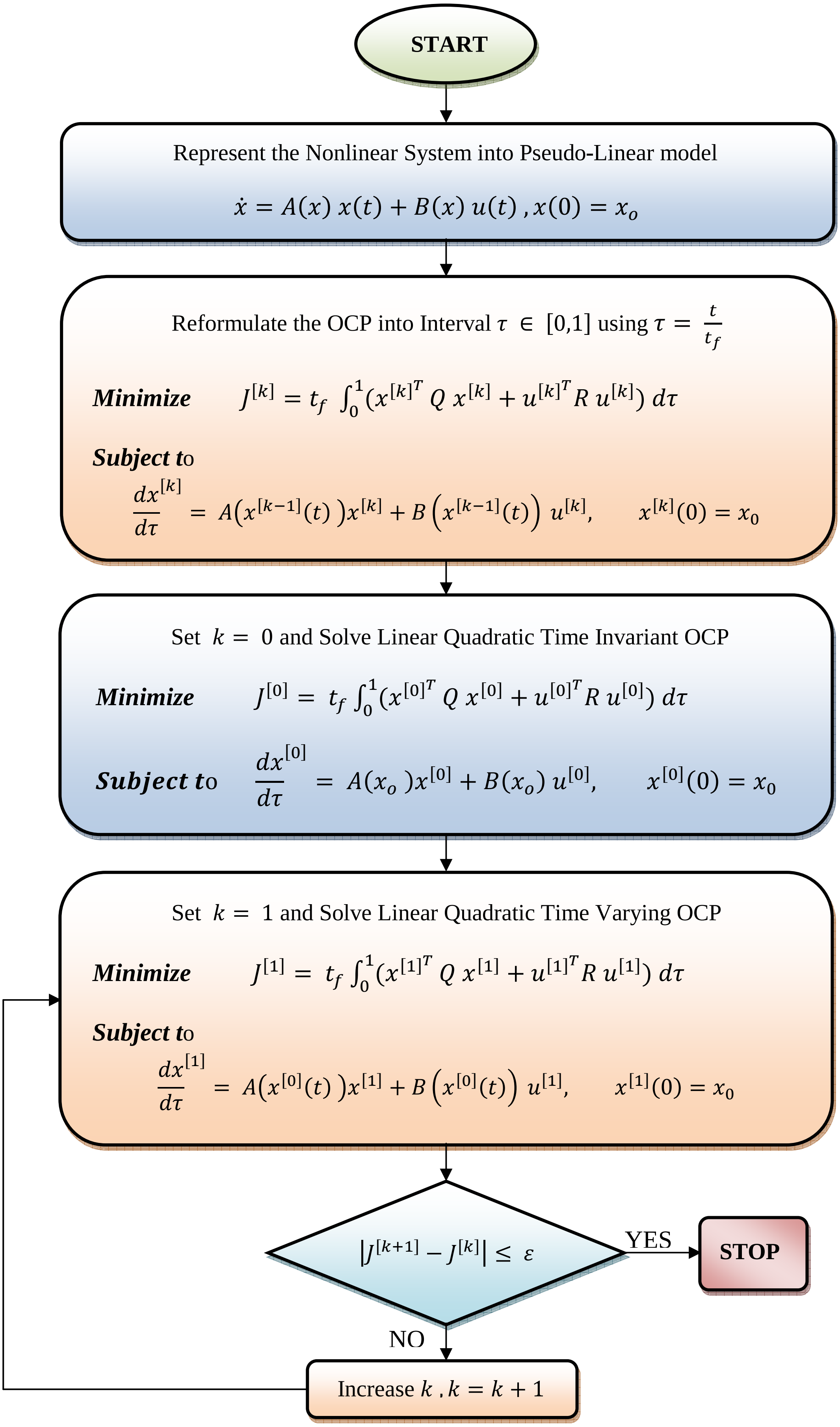

Figure (5.1): Flow Chart for Solving Nonlinear OCP Using Iterative Technique

5.4 Numerical Examples

5.4.1 Example (1)

Find the optimal control $u^*(t)$ for Van der Pol problem that minimizes the following performance index

$$J = \frac{1}{2}\int_0^5 (x_1^2 + x_2^2 + u^2)dt$$

Subject to nonlinear equality constraints and initial conditions as follows

$$\dot{x}_1 = x_2 \qquad , x_1(0) = 1$$

$$\dot{x}_2 = -x_1 + x_2 - x_1^2 x_2 + u \quad , x_2(0) = 0$$

This problem was solved by several researchers and by different methods. Jaddu [1] solved the problem using state parameterization via Chebyshev polynomials and quasilinearization technique and J was found to be 1.433487. Bullock and Franklin [28] solved the same problem using the second variation method and J was found to be 1.433508 . Bashein and Enns [29] using quasilinearization and discretization and J was found to be 1.438097. Majdalawi [15] using state parameterization via Legendre polynomials combined with iterative technique and J was found to be 1.449395 .

To solve this OCP using the proposed algorithm, we need to use the described procedures in flow chart of Figure (5.1) as follows

Step (1): rewrite the OCP in Pseudo-Linear form as follows

$$\dot{x}_1 = x_2 \qquad , x_1(0) = 1$$

$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2 + u \quad , x_2(0) = 0$$

Step (2): reformulate the OCP into Interval $\tau \in [0,1]$ using $\tau = \frac{t}{t_f} = \frac{t}{5}$, we get

Minimize

$$J^{[k]} = \frac{5}{2}\int_0^1 \left(\left(x_1^{[k]}\right)^2 + \left(x_2^{[k]}\right)^2 + \left(u^{[k]}\right)^2\right) d\tau$$

Subject to

$$\frac{dx_1{}^{[k]}}{d\tau} = 5x_2{}^{[k]} \qquad , x_1{}^{[k]}(0) = 1$$

$$\frac{dx_2{}^{[k]}}{d\tau} = 5\left[-x_1{}^{[k]} + \left(1 - \left(x_1{}^{[k-1]}\right)^2\right)x_2{}^{[k]} + u^{[k]}\right] \quad , x_2{}^{[k]}(0) = 0$$

Step (3): set $k = 0$ and solve linear quadratic time invariant OCP of the following form

$$\frac{dx_1^{[0]}}{d\tau} = 5x_2^{[0]} \qquad , x_1^{[0]}(0) = 1$$

$$\frac{dx_2^{[0]}}{d\tau} = -5x_1^{[0]} + 5u^{[0]} \quad , x_2^{[0]}(0) = 0$$

These state equations can be written in matrix form as follows

$$\begin{bmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \begin{bmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} [u]$$

This is the iteration zero and for $K = 2$ and $M = 4$, we get the optimal value $J = 0.9537225048$. The optimal state and control trajectories for $k = 0$ are shown in Figure (5.2). These trajectories are close to the exact trajectories.

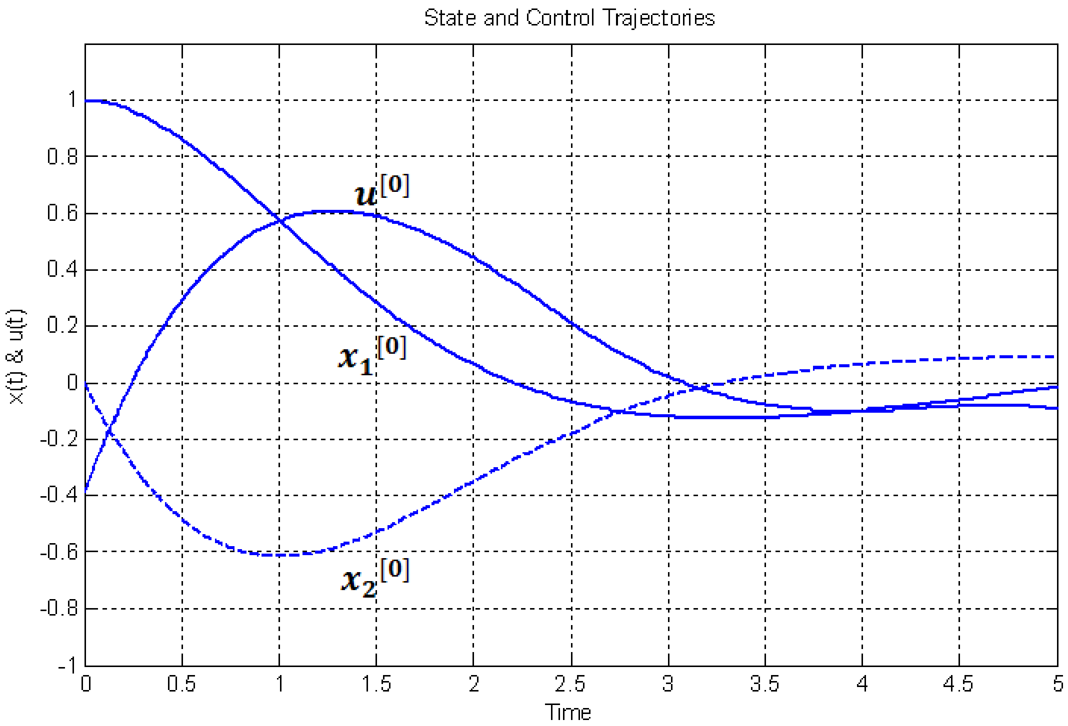

Figure (5.2): Optimal State and Control Trajectories at $k = 0$

Step (4): set $k = 1$ and solve linear quadratic time varying OCP as follows

$$\frac{dx_1^{[1]}}{d\tau} = 5x_2^{[1]} \qquad , x_1^{[1]}(0) = 1$$

$$\frac{dx_2^{[1]}}{d\tau} = -5x_1^{[1]} + 5\left(1 - \left(x_1^{[0]}\right)^2\right) x_2^{[1]} + 5u^{[1]} \quad , x_2^{[1]}(0) = 0$$

These state equations can be written in matrix form as follows

$$\begin{bmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -5 & 5\left(1-\left(x_1^{[0]}\right)^2\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \begin{bmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} [u]$$

This is the first iteration and using quadratic programming at $K = 2$ and $M = 4$, we get the optimal value $J = 1.549716935$.

Step (5): increase k and record the optimal values in Table (5.1)

To show applicability and effectiveness of our proposed algorithm, we compare our results with other researchers as shown in Table (5.2).

Table (5.1): Optimal Values for Example (1) for $K = 2$ and $M = 4$

Iteration ***k***	***J***
0	0.9537225048
1	1.549716935
2	1.476606108
3	1.457374586
4	1.449939834
5	1.449039711

Table (5.2): Comparison between Used Methods for Van der Pol Problem

Source	**J**
Jaddu [1]	1.433487
Bullock and Franklin [28]	1.433508
Bashein and Enns [29]	1.438097
Majdalawi [15]	1.449395
This Research	1.449039

The optimal state and control trajectories of Van der Pol problem at $k = 5$ are shown in Figure (5.3). These trajectories are very close to the exact trajectories.

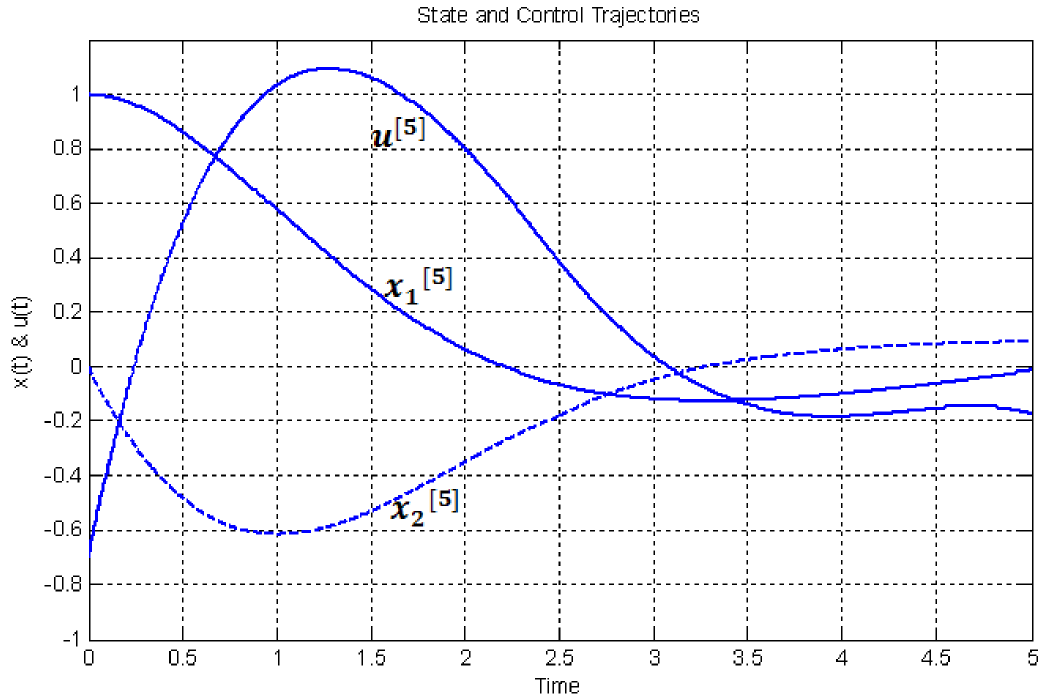

Figure (5.3): Optimal State and Control Trajectories at $k = 5$

5.4.2 Example (2)

Find the optimal control $u^*(t)$ for Rayleigh problem that minimizes the following performance index

$$J = \int_0^{2.5} (x_1^2 + u^2)dt$$

Subject to nonlinear equality constraints and initial conditions as follows

$$\dot{x}_1 = x_2 \qquad , x_1(0) = -5$$

$$\dot{x}_2 = -x_1 + 1.4x_2 - 0.14x_2^3 + 4u \quad , x_2(0) = -5$$

This problem was solved by several researchers and by different methods. Jaddu [1] solved the problem using state parameterization via Chebyshev polynomials and five quasilinearization iterations and J was found to be 29.4022. Nedeljkovic [30] found J to be 29.419 using three different algorithms which based on the first order Riccati equation. Sirisena [31] found J to be 29.451 using a piecewise polynomials parameterization.

To solve this problem, we will follow the same procedures as shown in flow chart in Figure (5.1).

Step (1): rewrite the OCP in Pseudo-Linear form as follows

$$\dot{x}_1 = x_2 \qquad , x_1(0) = -5$$

$$\dot{x}_2 = -x_1 + (1.4 - 0.14x_2^2)x_2 + 4u \quad , x_2(0) = -5$$

Step (2): reformulate the OCP into Interval $\tau \in [0,1]$ using $\tau = \frac{t}{t_f} = \frac{t}{2.5}$, we get

Minimize

$$J^{[k]} = \frac{5}{2}\int_0^1 \left(\left(x_1^{[k]}\right)^2 + \left(u^{[k]}\right)^2\right) d\tau$$

Subject to

$$\frac{dx_1{}^{[k]}}{d\tau} = \frac{5}{2}x_2{}^{[k]} \qquad , x_1{}^{[k]}(0) = -5$$

$$\frac{dx_2{}^{[k]}}{d\tau} = \frac{5}{2}\left[-x_1{}^{[k]} + \left(1.4 - 0.14\left(x_2{}^{[k-1]}\right)^2\right)x_2{}^{[k]} + 4u^{[k]}\right] \quad , x_2{}^{[k]}(0) = -5$$

Step (3): set $k = 0$ and solve linear quadratic time invariant OCP of the following form

$$\frac{dx_1{}^{[0]}}{d\tau} = \frac{5}{2}x_2{}^{[0]} \qquad , x_1{}^{[0]}(0) = -5$$

$$\frac{dx_2{}^{[0]}}{d\tau} = \frac{5}{2}\left[-x_1{}^{[0]} + 4u^{[0]}\right] \quad , x_2{}^{[0]}(0) = -5$$

These state equations can be written in matrix form as follows

$$\begin{bmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 & 2.5 \\ -2.5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \begin{bmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}[u]$$

Using quadratic programming to solve this OCP at $K = 2$ and $M = 4$, we get the optimal value $J = 29.63689449$. The optimal state and control trajectories at $k = 0$ are shown in Figure (5.4).

Step (4): set $k = 1$ and solve linear quadratic time varying OCP as follows

$$\frac{dx_1{}^{[1]}}{d\tau} = \frac{5}{2}x_2{}^{[1]} \qquad , x_1{}^{[1]}(0) = -5$$

$$\frac{dx_2{}^{[1]}}{d\tau} = \frac{5}{2}\left[-x_1{}^{[1]} + \left(1.4 - 0.14\left(x_2{}^{[0]}\right)^2\right)x_2{}^{[1]} + 4u^{[1]}\right] \quad , x_2{}^{[1]}(0) = -5$$

These state equations can be written in matrix form as follows

$$\begin{bmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 & 2.5 \\ -2.5 & 2.5\left(1.4 - 0.14\left(x_2^{[0]}\right)^2\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \begin{bmatrix} \frac{dx_1}{d\tau} \\ \frac{dx_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix} [u]$$

Solving the first iteration quadratic programming problem at $K = 2$ and $M = 4$, we get the optimal value $J = 29.54820588$.

Step (5): increase k and record the optimal values in Table (5.3)

Table (5.3): Optimal Values for Rayleigh problem at $K = 2$ and $M = 4$

Iteration ***k***	***J***
1	29.54820588
2	29.45927227
3	29.44150164
4	29.43936952
5	29.43439489

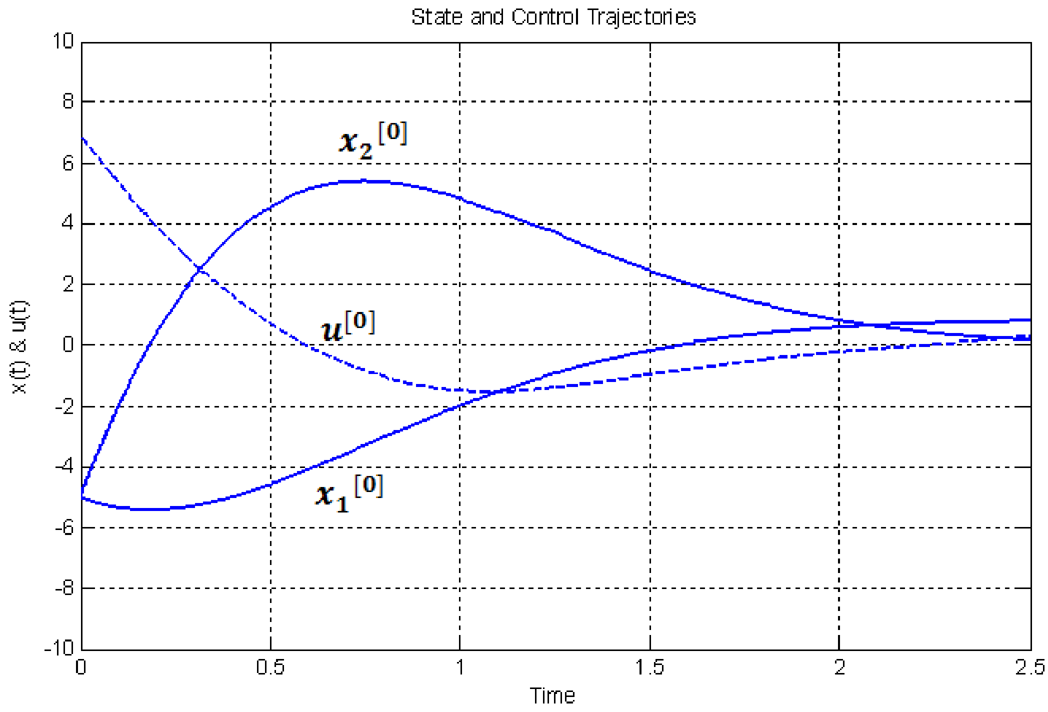

Figure (5.4): Optimal State and Control Trajectories at $k = 0$ for Rayleigh problem

To show effectiveness of the proposed method, we compare our result with other researchers as shown in Table (5.4).

Table (5.4): Comparison between Used Methods for Rayleigh Problem

Source	**J**
Jaddu [1]	29.4022
Nedeljkovic [30]	29.419
Sirisena [31]	29.451
This Research	29.4393

Chapter Six

Conclusion and Future Work

6.1 Conclusion

In this thesis, we proposed a numerical method to solve the nonlinear quadratic optimal control problem. However, the proposed method is also suitable for solving linear time invariant/ variant quadratic optimal control problems. The proposed method depends on state-control parameterization via Legendre scaling function and the iterative technique in which the nonlinear state equations are replaced by a sequence of linear time-varying state equations. In addition, we choose Legendre scaling function which has many advantages over the other orthogonal polynomials and functions.

Our proposed method, directly converts the optimal control problem into a quadratic programming problem. In which, the complex nonlinear quadratic optimal control problem is converted to a sequence of a quadratic linear optimal control problems that are much easier to solve.

The applicability and effectiveness of the proposed method have been proven through solving many numerical examples and by comparing our results with other researchers were used different orthogonal polynomials and functions.

6.2 Future Work

Searching in the field of wavelets functions usages in approximation and numerical methods is very rich and new. Moreover, the work in this thesis can be extended in many ways. Firstly, solving nonlinear optimal control problem with inequality constraints. Secondly, using operational matrix of differentiation instead of operational matrix of integration for Legendre scaling function.

References

[1] H. Jaddu, “Numerical Methods for Solving Optimal Control Problems Using Chebyshev Polynomials”, PhD Thesis, JAIST, Japan, 1998.

[2] M. Razzaghi and G. Elnagar, “Linear Quadratic Optimal Control Problems via Shifted Legendre State Parameterization”, *Int. J. Systems Science*, Vol. 25, pp. 393-399, 1994.

[3] C.Y. Yang and C. K. Chen, “Analysis and Optimal Control of Time Varying Systems via Fourier Series”, *Int. J. Systems Science*, Vol. 25, pp. 1663-1678, 1994.

[4] C. F. Chen and C. H. Hsiao, “Design of Piecewise Constant Gains for Optimal Control via Walsh Functions”, *IEEE Trans. Automatic Control*, Vol. 20, pp. 596-603, 1975.

[5] M. Tomas-Rodriguez. “Iteration Technique for Nonlinear Systems”: Applications to Control, PHD Thesis, The Sheffield University. Sheffield. United Kingdom, 2005.

[6] M. Tomas-Rodriguez, S.P. Banks, “An Iterative approach to Eignvalue Assignment for Nonlinear Systems”, *Proceedings of the 45th IEEE Conference on Decision & Control*, pp. 1-6, 2006.

[7] M. Tomas-Rodriguez, S. P. Banks and M.U. Salamci, “Sliding Mode Control for Nonlinear Systems: An Iterative Approach”, *Proceedings of the 45th IEEE Conference on Decision & Control*, pp. 1-6, 2006.

[8] M. Razzaghi and S. Yousefi, “Legendre Wavelets Method for Constrained Optimal Control Problems”, *Math Meth. Appl. Sci.*, Vol. 25, pp. 529-539, 2002.

[9] M. Razzaghi and S. Yousefi, “Legendre Wavelets Method for the Solution of Nonlinear Problems in the Calculus of Variations”, *Mathematical and Computer Modeling*, Vol. 34, pp. 45-54, 2001.

[10] M. Razzaghi and S. Yousefi, “Legendre Wavelets Direct Method for Variational Problems”, *Mathematics and Computers in Simulation.*, Vol. 53, pp. 185-192, 2000.

[11] M. Dadkhah, M. T. Kajani and S. Mahdavi, “Numerical Solution of Nonlinear Fredholm-Volterra Integro-Differetial Equation Using Legendre Wavelets”, *Proceedings of the* 6^{th} *IMT-GT Conference on Mathematics, Statistics and its Applications*, Malaysia, 2010.

[12] E. Babolian and F. Fattahzadeh, “Numerical Solution of Differential Equations by Using Chebyshev Wavelet Operational Matrix of Integration”, *Applied Mathematics and Computation*, Vol. 188, pp. 417-426, 2007.

[13] H. Jaddu and M. Vlach, “Wavelet-Based Approach to Optimizing Linear Systems”, *Japan Advanced Institute of Science and Technology*, Ishikawa 923-1292, Japan, 2006.

[14] H. Jaddu, “Optimal Control of Time-Varying Linear Systems Using Wavelets”, *Japan Advanced Institute of Science and Technology*, Ishikawa 923-1292, Japan, 2006.

[15] A. Majdalawi, “An Iterative Technique for Solving Nonlinear Quadratic Optimal Control Problem Using Orthogonal Functions”, MSC Thesis, Al-Quds University, Palestine, 2010.

[16] P. N. Paraskevopoulos, “Legendre Series Approach to Identification and Analysis of Linear Systems”, *IEEE Trans. Automat. Cont.*, VOL. AC-30, NO. 6, pp. 585-589, 1985.

[17] K. Teo, C. Goh and K. Wong, *A Unified Computational Approach to Optimal Control Problems*, Longman Scientific & Technical, England, 1991.

[18] H. C. Hsieh, “Synthesis of Adaptive Control Systems by Function Space Methods”, *Advances in Control Systems*, Ed: C. T. Leondes, Vol. 2, pp. 117-208, New York, Academic Press, 1965.

[19] C. P. Neuman and A. Sen, “A Suboptimal Control Algorithm for Constrained Problems Using Cubic Splines”, *Automatica*, Vol. 9, pp. 601-613, 1973.

[20] J. Vlassenbroeck, “A Chebyshev Polynomial Method for Optimal Control with State Constraints”, *Automatica*, Vol. 24, pp. 499-506, 1988.

[21] M. Razzaghi, “Fourier Series Direct Method for Variational Problems”, *International Journal of Control*, Vol. 48, pp. 887-895, 1988.

[22] J. W. Brewer, ”Kronecker Product and Matrix Calculus in System Theory”, *IEEE Transactions on Circuits and Systems*, Vol. 25, No. 9, 1978.

[23] G. Elnagar, ”State-Control Spectral Chebyshev Parameterization for Linearly Constrained Quadratic Optimal Control Problems”, *Journal of Computational and Applied Mathematics*”, Vol. 79, pp. 19-40, 1997.

[24] M. Tomas, C. N. Hernandez and S.P. Banks, “Parametric Approach to Optimal Nonlinear Control Problem Using Orthogonal Expansions”, *IFAC*, 2005.

[25] X. Zheng and X. Yang, “Techniques for Solving Integral and Differential Equations by Legendre Wavelets”, *International Journal of Systems Science*, Vol. 40, No. 11, pp. 1127-1137, 2009.

[26] H. Jaddu, “Direct Solution of Nonlinear Optimal Control Problems Using Quasilinearzation and Chebyshev Polynomials”, *Journal of the Franklin Institute*, Vol. 339, pp. 479-498, 2002.

[27] H. K. Khalil, *Nonlinear Systems*, *Prentice Hall*, Chapter three, pp. 87-95, third edition, 2002.

[28] T. Bullock and G. Franklin, “A Second Order Feedback Method for Optimal Control Computations”, *IEEE Trans. Automat. Cont.*, Vol. 12, pp. 666-673, 1967.

[29] G. Bashein and M. Enns, “Computation of Optimal Control by A Method Combining Quasi-linearization and Quadratic Programming”, *Int. J. Control*, Vol. 16, pp. 177-187,1972.

[30] N. B. Nedeljkovic, “New Algorithm for Unconstrained Nonlinear Optimal Control Problems”, *IEEE Trans. Automat. Cont.*, 26, 868-884, 1981.

[31] H. R. Sirisena, “Computation of Optimal Control Using A Piecewise Polynomials Parameterization”, *IEEE Trans. Automat. Cont.*, AC-18, 409-411, 1973.